THE INVERSE Math 21b, O. Knill

INVERTIBLE TRANSFORMATIONS. A map T from X to Y is **invertible** if there is for every $y \in Y$ a **unique** point $x \in X$ such that $T(x) = y$.

EXAMPLES.
1) $T(x) = x^3$ is invertible from $X = \mathbf{R}$ to $X = Y$.
2) $T(x) = x^2$ is not invertible from $X = \mathbf{R}$ to $X = Y$.
3) $T(x, y) = (x^2 + 3x - y, x)$ is invertible from $X = \mathbf{R}^2$ to $Y = \mathbf{R}^2$.
4) $T(\vec{x}) = Ax$ linear and $\mathrm{rref}(A)$ has an empty row, then T is not invertible.
5) If $T(\vec{x}) = Ax$ is linear and $\mathrm{rref}(A) = 1_n$, then T is invertible.

INVERSE OF LINEAR TRANSFORMATION. If A is a $n \times n$ matrix and $T : \vec{x} \mapsto Ax$ has an inverse S, then S is linear. The matrix A^{-1} belonging to $S = T^{-1}$ is called the **inverse matrix** of A.

First proof: check that S is linear using the characterization $S(\vec{a} + \vec{b}) = S(\vec{a}) + S(\vec{b}), S(\lambda\vec{a}) = \lambda S(\vec{a})$ of linearity.
Second proof: construct the inverse using Gauss-Jordan elimination.

FINDING THE INVERSE. Let 1_n be the $n \times n$ identity matrix. Start with $[A|1_n]$ and perform Gauss-Jordan elimination. Then

$$\mathrm{rref}([A|1_n]) = [1_n|A^{-1}]$$

Proof. The elimination process actually solves $A\vec{x} = \vec{e}_i$ simultaneously. This leads to solutions $\vec{v}_i$ which are the columns of the inverse matrix A^{-1} because $A^{-1}\vec{e}_i = \vec{v}_i$.

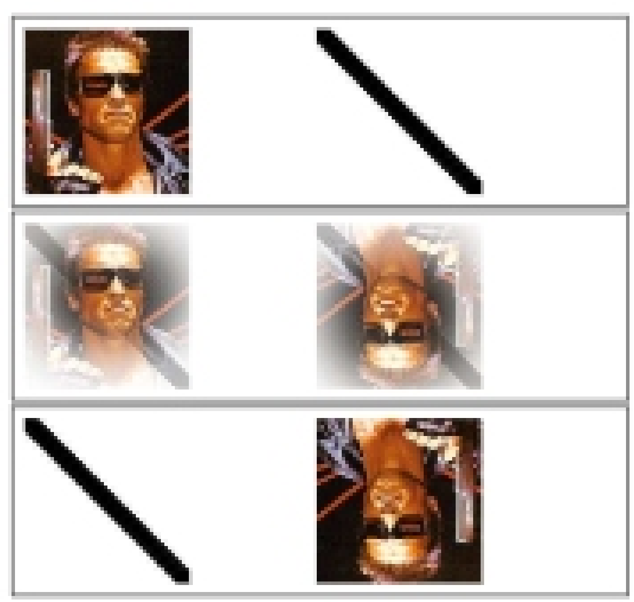
EXAMPLE. Find the inverse of $A = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 2 & 6 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right] \qquad [\ A \ | \ 1_2\ ]$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1/2 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right] \qquad [\ .... \ | \ ...\ ]$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1/2 & 0 \\ 0 & 1 & -1/2 & 1 \end{array}\right] \qquad [\ .... \ | \ ...\ ]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1/2 & 1 \end{array}\right] \qquad [\ 1_2 \ | \ A^{-1}\ ]$$



The inverse is $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1/2 & 1 \end{bmatrix}$.

THE INVERSE OF LINEAR MAPS $R^2 \mapsto R^2$:

If $ad - bc \neq 0$, the inverse of a linear transformation $\vec{x} \mapsto Ax$ with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by the matrix $A^{-1} = \begin{bmatrix} d & -b \\ -d & a \end{bmatrix} / (ad - bc)$.

SHEAR:
$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

DIAGONAL:

$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$

REFLECTION:

$A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$ $\quad A^{-1} = A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$

ROTATION:

$A = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(-\alpha) \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$

ROTATION-DILATION:

$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} a/r^2 & b/r^2 \\ -b/r^2 & a/r^2 \end{bmatrix}$, $r^2 = a^2 + b^2$

BOOST:

$A = \begin{bmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & \cosh(\alpha) \end{bmatrix}$

NONINVERTIBLE EXAMPLE. The projection $\vec{x} \mapsto A\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a non-invertible transformation.

MORE ON SHEARS. The shears $T(x,y) = (x + ay, y)$ or $T(x,y) = (x, y + ax)$ in $\mathbf{R}^2$ can be generalized. A shear is a linear transformation which fixes some line L through the origin and which has the property that $T(\vec{x}) - \vec{x}$ is parallel to L for all $\vec{x}$.

PROBLEM. $T(x,y) = (3x/2 + y/2, y/2 - x/2)$ is a shear along a line L. Find L.
SOLUTION. Solve the system $T(x,y) = (x,y)$. You find that the vector $(1,-1)$ is preserved.

MORE ON PROJECTIONS. A linear map T with the property that $T(T(x)) = T(x)$ is a projection. Examples: $T(\vec{x}) = (\vec{y} \cdot \vec{x})\vec{y}$ is a projection onto a line spanned by a unit vector $\vec{y}$.

WHERE DO PROJECTIONS APPEAR? CAD: describe 3D objects using projections. A photo of an image is a projection. Compression algorithms like JPG or MPG or MP3 use projections where the high frequencies are cut away.

MORE ON ROTATIONS. A linear map T which preserves the angle between two vectors and the length of each vector is called a **rotation**. Rotations form an important class of transformations and will be treated later in more detail. In two dimensions, every rotation is of the form $x \mapsto A(x)$ with $A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$.
An example of a rotations in three dimensions are $\vec{x} \mapsto Ax$, with $A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$. it is a rotation around the z axis.

MORE ON REFLECTIONS. Reflections are linear transformations different from the identity which are equal to their own inverse. Examples:
2D reflections at the origin: $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, **2D reflections at a line** $A = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$.
3D reflections at origin: $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. **3D reflections at a line** $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. By the way: in any dimensions, to a reflection at the line containing the unit vector $\vec{u}$ belongs the matrix $[A]_{ij} = 2(u_i u_j) - [1_n]_{ij}$, because $[B]_{ij} = u_i u_j$ is the matrix belonging to the projection onto the line.
The reflection at a line containg the unit vector $\vec{u} = [u_1, u_2, u_3]$ is $A = \begin{bmatrix} u_1^2 - 1 & u_1 u_2 & u_1 u_3 \\ u_2 u_1 & u_2^2 - 1 & u_2 u_3 \\ u_3 u_1 & u_3 u_2 & u_3^2 - 1 \end{bmatrix}$.
3D reflection at a plane $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.
Reflections are important symmetries in physics: T (time reflection), P (reflection at a mirror), C (change of charge) are reflections. It seems today that the composition of TCP is a fundamental symmetry in nature.