

3.2-Solving Systems of Linear Equations Using Matrices

The Gauss-Jordan Elimination Method

Start with a system of linear equations (this class solves up to a 3 x 3 system).

$$2x + 4y = 6$$

$$x + 3y = 4$$

1. Write the augmented matrix corresponding to the linear system.

$$\left[\begin{array}{cc|c} 2 & 4 & 6 \\ 1 & 3 & 4 \end{array} \right]$$

2. Perform the following row operations to solve this problem.

Row Operations examples:

1. Interchange any two rows.

$$\left[\begin{array}{ccc} 2 & -1 & 3 \\ 1 & 3 & 5 \end{array} \right] R_1 \leftrightarrow R_2 \quad \left[\begin{array}{ccc} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array} \right]$$

2. Replace any row by a nonzero constant multiple of itself.

$$\left[\begin{array}{ccc} 2 & -1 & 3 \\ 4 & -2 & 8 \end{array} \right] \frac{-1}{2}R_2 \rightarrow R_2 \quad \left[\begin{array}{ccc} 2 & -1 & 3 \\ -2 & 1 & -4 \end{array} \right]$$

3. Replace any row by the sum of that row and a constant multiple of any other row.

$$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array} \right] -2R_1 + R_2 \rightarrow R_2 \quad \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -7 & -7 \end{array} \right]$$

First step to solving system using Gauss-Jordan

The sequence of row operations that transforms the augmented matrix into the equivalent matrix in which the 1st element is a 1

$$\left[\begin{array}{cc|c} 2 & 4 & 6 \\ 1 & 3 & 4 \end{array} \right] \text{ so the } 2 \text{ needs to be a } 1. \text{ You must use only the row operations from above.}$$

Handwritten solution:

$$\left[\begin{array}{cc|c} 2 & 4 & 6 \\ 1 & 3 & 4 \end{array} \right] \frac{1}{2}R_1 \rightarrow R_1 \quad \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & 3 & 4 \end{array} \right] -R_1 + R_2 \rightarrow R_2$$
$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \end{array} \right] -2R_2 + R_1 \rightarrow R_1 \quad \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array} \right] \quad \begin{array}{l} x=1 \\ y=1 \end{array}$$

Annotations: "pivot element" with an arrow pointing to the 1 in the top-left of the second matrix, and "pivot element" with a circle around the 1 in the bottom-left of the third matrix.

3.2-Solving Systems of Linear Equations Using Matrices

To change the column 1 $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ into the unit column $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we perform certain row operations

and we call this pivoting the matrix about the element 2. This is how you start all Gauss-Jordan problems.

Let's solve this problem:

$$\begin{bmatrix} 2 & 4 & | & 6 \\ 1 & 3 & | & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 1 \end{bmatrix}$$

Example 1:

Solve the system of linear equations using the Gauss-Jordan elimination method.

$$y - 8z = 9$$

$$x - 2y + 3z = -3$$

$$7y - 5z = 12$$

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} x & y & z & | & \\ 0 & 1 & -8 & | & 9 \\ 1 & -2 & 3 & | & -3 \\ 0 & 7 & -5 & | & 12 \end{bmatrix} \quad R_1 \leftrightarrow R_2 \quad \begin{bmatrix} 1 & -2 & 3 & | & -3 \\ 0 & 1 & -8 & | & 9 \\ 0 & 7 & -5 & | & 12 \end{bmatrix} \quad \begin{array}{l} 2R_2 + R_1 \rightarrow R_1 \\ -7R_2 + R_3 \rightarrow R_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -13 & | & 15 \\ 0 & 1 & -8 & | & 9 \\ 0 & 0 & 51 & | & -51 \end{bmatrix} \quad \frac{1}{51} R_3 \rightarrow R_3 \quad \begin{bmatrix} 1 & 0 & -13 & | & 15 \\ 0 & 1 & -8 & | & 9 \\ 0 & 0 & 1 & | & -1 \end{bmatrix} \quad \begin{array}{l} 13R_3 + R_1 \rightarrow R_1 \\ 8R_3 + R_2 \rightarrow R_2 \end{array}$$

$$\begin{bmatrix} x & y & z & | & \\ 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & -1 \end{bmatrix} \quad \begin{array}{l} x=2 \\ y=1 \\ z=-1 \end{array}$$

3.2-Solving Systems of Linear Equations Using Matrices

Example 2: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$\begin{array}{l}
 x + 2y + 3z = 7 \\
 2x + 4y - 6z = 38 \\
 3x - 4y + 4z = -19
 \end{array}
 \begin{array}{l}
 R_1 \\
 R_2 \\
 R_3
 \end{array}
 \left[\begin{array}{ccc|c}
 1 & 2 & 3 & 7 \\
 2 & 4 & -6 & 38 \\
 3 & -4 & 4 & -19
 \end{array} \right]
 \begin{array}{l}
 -2R_1 + R_2 \rightarrow R_2 \\
 -3R_1 + R_3 \rightarrow R_3
 \end{array}
 \left[\begin{array}{ccc|c}
 1 & 2 & 3 & 7 \\
 0 & 0 & -12 & 24 \\
 0 & -10 & -5 & -40
 \end{array} \right]
 \begin{array}{l}
 R_2 \leftrightarrow R_3
 \end{array}$$

$$\left[\begin{array}{ccc|c}
 1 & 2 & 3 & 7 \\
 0 & -10 & -5 & -40 \\
 0 & 0 & -12 & 24
 \end{array} \right]
 \begin{array}{l}
 -\frac{1}{10}R_2 \rightarrow R_2
 \end{array}
 \left[\begin{array}{ccc|c}
 1 & 2 & 3 & 7 \\
 0 & 1 & \frac{1}{2} & 4 \\
 0 & 0 & -12 & 24
 \end{array} \right]
 \begin{array}{l}
 -2R_2 + R_1 \rightarrow R_1
 \end{array}$$

$$\left[\begin{array}{ccc|c}
 1 & 0 & 2 & -1 \\
 0 & 1 & \frac{1}{2} & 4 \\
 0 & 0 & -12 & 24
 \end{array} \right]
 \begin{array}{l}
 -\frac{1}{12}R_3 \rightarrow R_3
 \end{array}
 \left[\begin{array}{ccc|c}
 1 & 0 & 2 & -1 \\
 0 & 1 & \frac{1}{2} & 4 \\
 0 & 0 & 1 & -2
 \end{array} \right]
 \begin{array}{l}
 -2R_3 + R_1 \rightarrow R_1 \\
 -\frac{1}{2}R_3 + R_2 \rightarrow R_2
 \end{array}$$

$$\begin{array}{c}
 x \quad y \quad z \\
 \left[\begin{array}{ccc|c}
 1 & 0 & 0 & 3 \\
 0 & 1 & 0 & 5 \\
 0 & 0 & 1 & -2
 \end{array} \right]
 \end{array}
 \begin{array}{l}
 x=3 \\
 y=5 \\
 z=-2
 \end{array}$$

An $m \times n$ augmented matrix is in row-reduced form if it satisfies the following conditions:

- Each row consisting entirely of zeros lies below any other row having nonzero entries.

$$\begin{array}{c}
 x \quad y \\
 \left[\begin{array}{cc|c}
 1 & 0 & -3 \\
 0 & 0 & 0 \\
 0 & 1 & -2
 \end{array} \right]
 \end{array}
 \begin{array}{l}
 \text{the correct row-reduced form} \\
 \left[\begin{array}{cc|c}
 1 & 0 & -3 \\
 0 & 1 & -2 \\
 0 & 0 & 0
 \end{array} \right]
 \end{array}$$

- The first nonzero entry in each row is 1 (called a leading 1).

$$\begin{array}{c}
 \left[\begin{array}{ccc|c}
 1 & 0 & 0 & -1 \\
 0 & 2 & 0 & 3 \\
 0 & 0 & 1 & -5
 \end{array} \right]
 \end{array}
 \begin{array}{l}
 \text{the correct row-reduced form} \\
 \left[\begin{array}{ccc|c}
 1 & 0 & 0 & -1 \\
 0 & 1 & 0 & \frac{3}{2} \\
 0 & 0 & 1 & -5
 \end{array} \right]
 \end{array}$$