

Math 2250-1

Wednesday Dec. 10

69.4 Phase portraits for mechanical systems

rigid-rod pendulum:

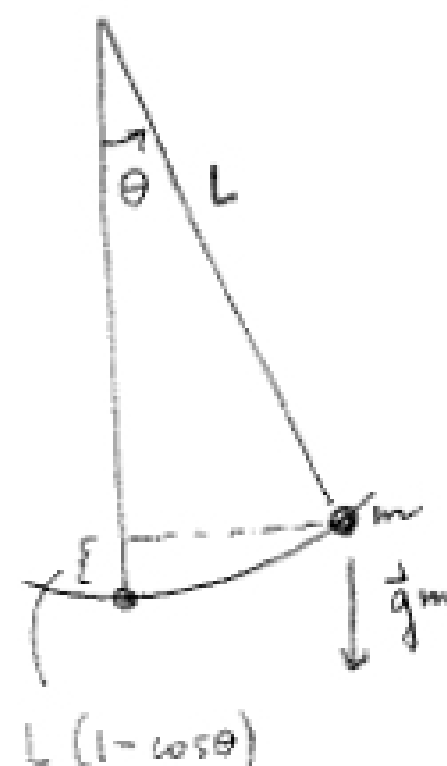
$$(1) \quad \theta''(t) + \frac{g}{L} \sin \theta = 0$$

We derived this DE using
KE + PE = constant

$$\frac{1}{2} m L^2 (\theta')^2 + mgL(1 - \cos \theta) = \text{const}$$

$$(2) \quad \frac{1}{2} L (\theta')^2 + g(1 - \cos \theta) = \tilde{\text{const}}$$

(and we took $\frac{d}{dt}(\cdot)$ to get (1))



For

$$x = \theta(t)$$

$$y = \theta'(t)$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -\frac{g}{L} \sin x \end{bmatrix}$$

equil sol'ns $y = 0$

$$\sin x = 0: \quad x = k\pi, \quad k \in \mathbb{Z} \quad (0, \pm 1, \pm 2, \dots)$$

Linearization:

$$J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} \cos x & 0 \end{bmatrix}$$

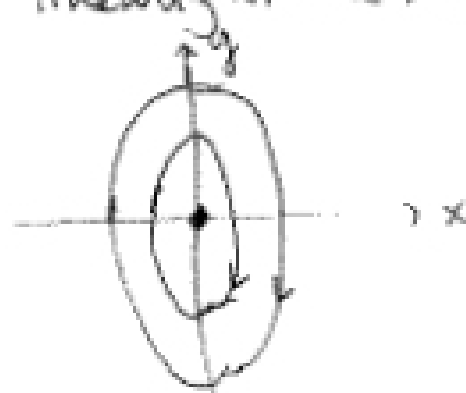
if $x = k\pi$ k even

$$J = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix}$$

$$|J - \lambda I| = \lambda^2 + \left(\frac{g}{L}\right) = 0$$

$$\lambda = \pm i\sqrt{\frac{g}{L}}$$

linearization has stable center



(why clockwise?)

if $x = k\pi$, k odd

$$J = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & 0 \end{bmatrix}$$

$$|J - \lambda I| = \lambda^2 - \frac{g}{L} = 0$$

$$\lambda = \pm \sqrt{\frac{g}{L}} \quad \text{saddle}$$

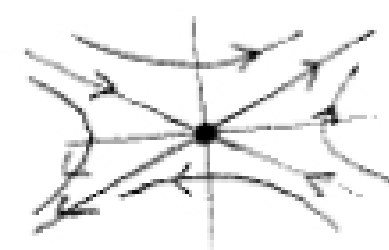
$$\lambda = \sqrt{\frac{g}{L}}$$

$$\lambda = -\sqrt{\frac{g}{L}}$$

$$\begin{bmatrix} -\sqrt{\frac{g}{L}} & 1 \\ \frac{g}{L} & -\sqrt{\frac{g}{L}} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} -1 \\ \sqrt{\frac{g}{L}} \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 1 \\ \sqrt{\frac{g}{L}} \end{bmatrix}$$



Final exam

Tuesday December 16 8-10 a.m.

here (ST 104)...

Practice exam will be posted.

Friday will be a review of entire course

Saturday review session (practice exam) 11-1.
room to be announced.

①

fill in?

2

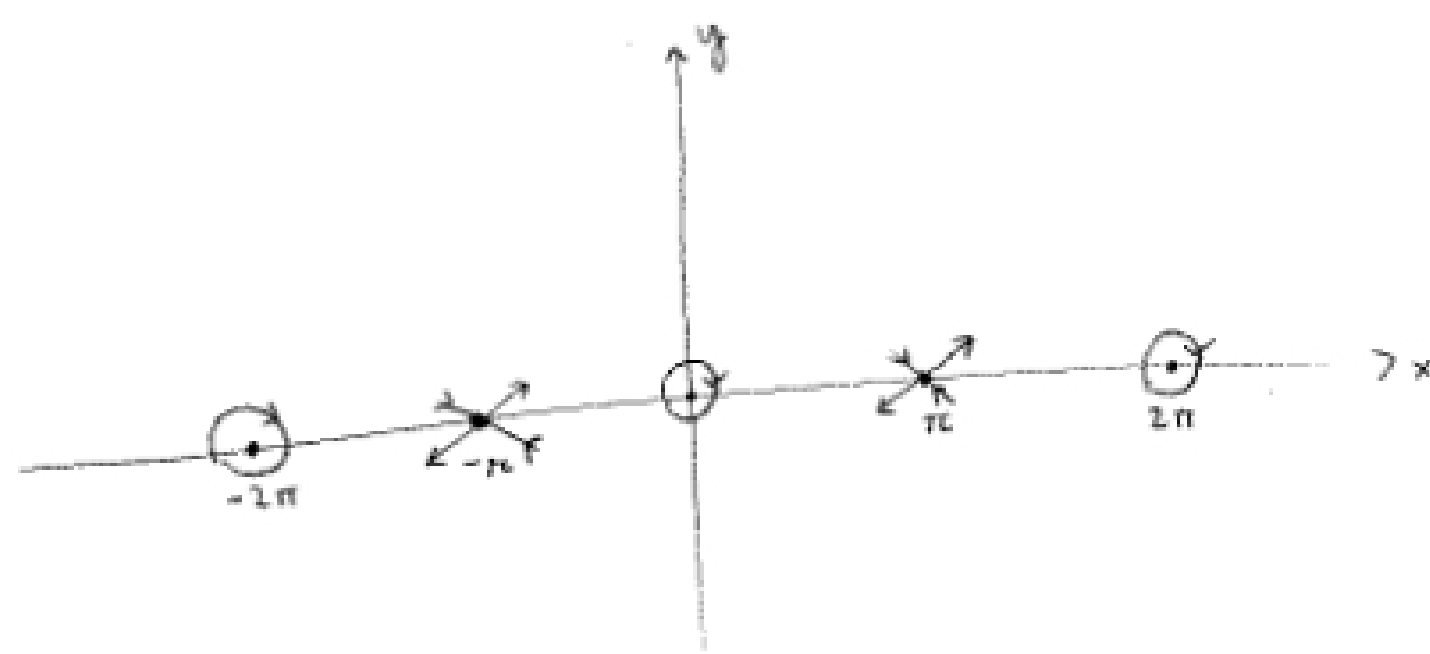
Hint: the trajectories must follow the level curves of the (scaled) total energy function (2), on page 1, i.e.

$$\frac{1}{2} L \dot{y}^2 + g(1 - \cos x) = \text{const} \quad \leftarrow \text{RHS is } 2\pi\text{-periodic in } x.$$

notice this function of (x, y)

attains its minimum value of zero, at all $(x, y) = (k\pi, 0)$
with k even
(so $\cos x = 1$)

and the Hessian matrix of this function is diagonal with positive diagonal entries, at these points, so the nearby trajectories for our system are (nearly) ellipses, for the non-linear system, so $(k\pi, 0)$ is stable center for the nonlinear problem, when k is even!



Add damping:

$$\ddot{\theta} + c\dot{\theta} + \frac{g}{L}\sin(\theta) = 0$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -\frac{g}{L}\sin x - cy \end{bmatrix}$$

same equilibria (iff $y=0$ & $\sin x=0$)

$$J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos x & -c \end{bmatrix}$$

$$|J - \lambda I| = -\lambda(-\lambda - c) + \frac{g}{L}\cos x$$

$$= \lambda^2 + c\lambda \pm \frac{g}{L} \quad \begin{matrix} (+\frac{g}{L} \text{ if } x=k\pi \\ \text{ } \\ -\frac{g}{L} \text{ if } k \text{ odd} \end{matrix}$$

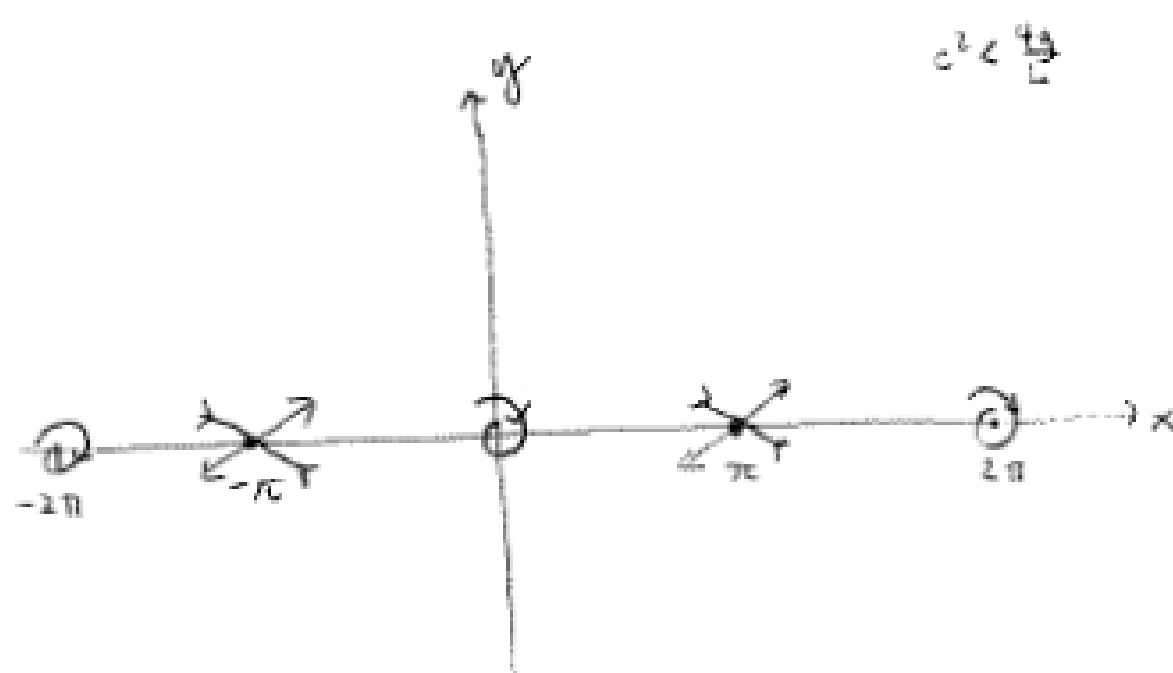
$$\text{roots } \lambda = \frac{-c \pm \sqrt{c^2 \mp \frac{4g}{L}}}{2}$$

k odd $\Rightarrow$ saddle

k even, $c^2 < \frac{4g}{L} \Rightarrow$ stable spiral (underdamped)

$c^2 > \frac{4g}{L} \Rightarrow$ stable node (overdamped)

Fill in?



pictures for page 1-2:

