

## 4.4 Undetermined Coefficients

The **method of undetermined coefficients** applies to solve differential equations

$$(1) \quad ay'' + by' + cy = f(x).$$

**Restrictions:** The symbols  $a, b, c$  are constant,  $a \neq 0$ . The nonhomogeneous term  $f(x)$  is a sum of terms, each of which is one the following forms, called **atoms**:

$$(2) \quad \begin{array}{ll} p(x) & \text{polynomial,} \\ p(x)e^{kx} & \text{polynomial} \times \text{exponential,} \\ p(x)e^{kx} \cos(mx) & \text{polynomial} \times \text{exponential} \times \text{cosine,} \\ p(x)e^{kx} \sin(mx) & \text{polynomial} \times \text{exponential} \times \text{sine.} \end{array}$$

The polynomial  $p(x)$  can be a constant. Symbols  $k$  and  $m \geq 0$  are constants. The trigonometric terms may appear without an exponential, e.g.,  $(1 + 2x)e^{0x} \sin 3x$  is normally written  $(1 + 2x) \sin 3x$ . The method's importance is argued from its direct applicability to second order differential equations in mechanics and circuit theory.

*Included* as possible functions  $f$  in (1) are  $\sinh x$  and  $\cos^3 x$ , due to identities from algebra and trigonometry. *Specifically excluded* are  $\ln|x|$ ,  $|x|$ ,  $e^{x^2}$  and fractions like  $x/(1 + x^2)$ .

**Superposition**  $y = y_h + y_p$  allows us to solve equation (1) in two stages: (a) Apply the linear equation **recipe** to find  $y_h$ ; (b) Apply **the method of undetermined coefficients** to find  $y_p$ . We expect to find two arbitrary constants  $c_1, c_2$  in the solution  $y_h$ , but in contrast, no arbitrary constants appear in  $y_p$ .

The **basic trial solution method**, which requires linear algebra, is presented on page 174. Readers should make an effort to learn this method, because literature normally omits details of the method, referencing only *the method of undetermined coefficients*. To enrich this basic method, we add a **library of special methods** for finding  $y_p$ , which includes Kümmer's method; see page ???. The library uses only college algebra and polynomial calculus. The trademark of the library method is the *absence of linear algebra, tables or special cases*, that can be found in other literature on the subject.

### The Algorithm for Undetermined Coefficients

A particular solution  $y_p$  of (1) will be expressed as a sum

$$y_p = y_1 + \cdots + y_n$$

where each  $y_k$  solves a related easily-solved differential equation.

The idea can be quickly communicated for  $n = 3$ . The superposition principle applied to the three equations

$$(3) \quad \begin{aligned} ay_1'' + by_1' + cy_1 &= f_1(x), \\ ay_2'' + by_2' + cy_2 &= f_2(x), \\ ay_3'' + by_3' + cy_3 &= f_3(x) \end{aligned}$$

shows that  $y = y_1 + y_2 + y_3$  is a solution of

$$(4) \quad ay'' + by' + cy = f_1 + f_2 + f_3.$$

If each equation in (3) is easily solved, then solving equation (4) is also easy: *add the three answers for the easily solved problems.*

To use the idea, it is necessary to start with  $f(x)$  and determine a decomposition  $f = f_1 + f_2 + f_3$  so that equations (3) are easily solved.

The process is called the **method of undetermined coefficients**. This method requires decomposing (1) into a number of **easily-solved equations**. For instance, if an easily-solved equation has forcing term  $f(x)$  equal to a polynomial, then a particular solution is found by substituting a **polynomial trial solution**

$$y = d_0 + d_1x + \cdots + d_m \frac{x^m}{m!}$$

with **undetermined coefficients**  $d_0, \dots, d_m$ . Undetermined coefficients are found by calculus and college algebra back-substitution.

**The Easily Solved Equations.** Each easily-solved equation is engineered to have right side in one of the four forms below, each of which is called an **atom**:

$$(5) \quad \begin{array}{ll} p(x) & \text{polynomial,} \\ p(x)e^{kx} & \text{polynomial} \times \text{exponential,} \\ p(x)e^{kx} \cos mx & \text{polynomial} \times \text{exponential} \times \text{cosine,} \\ p(x)e^{kx} \sin mx & \text{polynomial} \times \text{exponential} \times \text{sine.} \end{array}$$

To illustrate, consider

$$(6) \quad ay'' + by' + cy = x + xe^x + x^2 \sin x - \pi e^{2x} \cos x + x^3.$$

The right side is decomposed as follows, in order to define the easily solved equations (also called the **atomic equations**):

$$\begin{array}{ll} ay_1'' + by_1' + cy_1 = x + x^3 & \text{Polynomial.} \\ ay_2'' + by_2' + cy_2 = xe^x & \text{Polynomial} \times \text{exponential.} \\ ay_3'' + by_3' + cy_3 = x^2 \sin x & \text{Polynomial} \times \text{exponential} \times \text{sine.} \\ ay_4'' + by_4' + cy_4 = -\pi e^{2x} \cos x & \text{Polynomial} \times \text{exponential} \times \text{cosine.} \end{array}$$

There are  $n = 4$  equations. In the illustration,  $x^3$  is included with  $x$ , but it could have caused creation of a fifth equation. To decrease effort, minimize the number  $n$  of easily solved equations. *One final checkpoint:* the right sides of the  $n$  equations must add to the right side of (6).

## The Basic Trial Solution Method

Literature referencing *the method of undetermined coefficients* usually means the basic trial solution method. Readers are asked to spend enough time to understand the method's mechanics and intricacies.

Assume given a constant-coefficient second order differential equation  $ay'' + by' + cy = f(x)$ , with  $f$  an **atom**, as in (2). The method:

**Homogeneous solution.** Solve the homogeneous equation for  $y_h$  by the *recipe*. It contains arbitrary constants  $c_1, c_2$ .

**Initial trial solution.** Differentiate the atom  $f(x)$  repeatedly. Isolate independent functions whose linear combinations are the derivatives. Multiply them by **undetermined coefficients**  $d_1, d_2, \dots, d_k$  to define an initial trial solution.

**Fixup rule.** If the initial trial solution duplicates terms found in  $y_h$ , then multiply the trial solution by  $x$  repeatedly until it doesn't. The **final trial solution** is the modified expression.

**Substitute and evaluate.** Substitute the final trial solution into the nonhomogeneous differential equation. Match coefficients of the independent functions to obtain equations for the undetermined coefficients  $d_1, d_2, \dots, d_k$ . Solve the system.

**Report**  $y = y_h + y_p$ . Homogeneous solution  $y_h$  was reported above. Particular solution  $y_p$  is the final trial solution with the evaluated coefficients. Add to obtain the general solution  $y$ .

The algorithm actually works for sums of atoms, provided the fixup rule is applied to each individual atom. Linear algebra techniques are used to solve the system of equations in the 4th step. An answer check is prudent, because of many opportunities for arithmetic errors.

**Illustration.** Let's solve  $y'' - y = x + xe^x$ , verifying that  $y_h = c_1e^x + c_2e^{-x}$  and  $y_p = -x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x$ .

- **Homogeneous solution.** The characteristic equation  $r^2 - 1 = 0$  has roots  $r = \pm 1$ . Recipe **case 1** implies  $y_h = c_1e^x + c_2e^{-x}$ .
- **Initial trial solution.** The atoms of  $f = x + xe^x$  are  $f_1 = x$ ,  $f_2 = xe^x$ . Then  $f = f_1 + f_2$  and the easily-solved problems are  $y_1'' - y_1 = x$  and  $y_2'' - y_2 = xe^x$ . A particular solution is  $y_p = y_1 + y_2$ . Initial trial solutions, found by differentiation, involve the independent terms 1,  $x$  for  $y_1$  and  $e^x, xe^x$  for  $y_2$ . Then  $y_1 = d_1 + d_2x$ ,  $y_2 = d_3e^x + d_4xe^x$ . The undetermined coefficients are  $d_1, d_2, d_3, d_4$ .