1 Problems with the Schwarzschild Metric

The spacetime outside of a non-rotating star (or planet or whatever) of total mass m is described by the metric tensor

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{1}{1 - \frac{2m}{r}}dr^2 + r^2 d\Omega^2$$

This metric solves the vacuum Einstein equations and, according to Birkhoff's Theorem, is the only spherically symmetric metric that does. It obviously has a problem at $r = 2m$. One metric coefficient blows up while another goes to zero.

Consider a clock that is holding its position at a constant value of the radial coordinate. If it does this for an interval Δt of coordinate time, then the time elapsed on the clock will be

$$\Delta\tau = \Delta t\sqrt{1 - \frac{2m}{r}}$$

For r near $2m$ very little proper time will elapse on the clock even though a great deal of coordinate time elapses. This result tells us that the $t =$ constant hypersurfaces are failing to advance in time near $r = 2m$.

During the early days of general relativity, the problem with the Schwarzschild solution was regarded mostly as a curiosity of little consequence because, for normal astronomical objects such as the sun and the earth, the critical value of the radius is extremely small. For an object with the mass of the earth, the critical radius is about a centimeter. For the sun, it is a kilometer. The metric inside an ordinary star is not given by the Schwarzschild vacuum solution and is quite regular everywhere. For the external vacuum spacetime to extend to the critical radius, the sun would have to be compressed to a radius of a kilometer. Until 1939, (Oppenheimer and Volkoff, Oppenheimer and Snyder) no astrophysicist seriously believed that a mass equal to that of the sun could be compressed into an object only a kilometer in radius. Not too many believed it after that either.

From a mathematical point of view, the coordinate-independent nature of spacetime geometry was not well understood during the early days, so it was some time before someone considered the possibility that the $r = 2m$ singularity was simply the result of bad coordinates. One indication that this might be the case is that none of the scalar curvature invariants such as R, $R_\rho{}^\sigma{}_{\mu\nu}R_\sigma{}^{\rho\mu\nu}$ and invariant ratios of lightlike components of curvatures become infinite at $r = 2m$.

2 The Kruskal Extension

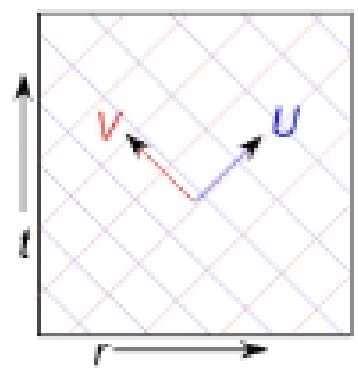
2.1 Orthogonal Surface Metric

Once it is realized that the $r = 2m$ singularity is really just a problem with the coordinates, it is fairly easy to fix. The angle coordinates θ, φ are fixed by the spherical symmetry group, so nothing could be wrong with them. That leaves the coordinates r, t and the metric

$$^2ds^2 = -fdt^2 + f^{-1}dr^2, \qquad f = 1 - \frac{2m}{r}$$

on the timelike two-surfaces orthogonal to the group orbits. We need to find new coordinates in which this metric looks more regular.

An obvious geometrical feature of this two dimensional metric is the light cone. On any such surface, it is always possible to find coordinates U, V such that curves at constant U are lightlike and so are curves at constant V. For example, in two dimensional Minkowski spacetime, one can take $U = t+r$, $V = t-r$ and obtain the metric in the form $^2ds^2 = -dUdV$.



In general, so long as the metric takes the form

$$^2ds^2 = -\Phi dUdV$$

then an interval with either $\Delta U = 0$ or $\Delta V = 0$ will be lightlike. Thus, we seek coordinates U, V such that

$$-fdt^2 + f^{-1}dr^2 = -\Phi dUdV$$

for some function Φ. Once we find such coordinates, we can construct space and time coordinates $u = U - V$, $v = U + V$ in which the light cones look exactly like the ones in Minkowski spacetime. If there is any coordinate system in which this metric tensor is regular, then this has to be the one.

2.2 Conditions on Advanced and Retarded Time Coordinates

Represent partial derivatives with respect to t, r by subscripts so that

$$dU = U_r dr + U_t dt, \qquad dV = V_r dr + V_t dt$$

and therefore

$$-fdt^2 + f^{-1}dr^2 = -\Phi\left(U_r dr + U_t dt\right)\left(V_r dr + V_t dt\right)$$

or

$$\Phi U_t V_t = f, \qquad \Phi U_r V_r = -f^{-1}$$
$$U_r V_t + U_t V_r = 0$$

Divide all of these equations by $U_r V_r$ and obtain the results

$$\frac{U_t}{U_r}\frac{V_t}{V_r} = -f^2, \qquad \frac{V_t}{V_r} + \frac{U_t}{U_r} = 0,$$

which can be solved for the two ratios in the form

$$\frac{U_t}{U_r} = f, \qquad \frac{V_t}{V_r} = -f.$$

Thus, if the coordinates exist, then they must satisfy these two conditions. Conversely, the argument can be reversed to show that solving these two conditions is enough to put the spacetime metric into the desired form where the conformal factor Φ can be found from the equation $\Phi U_t V_t = f$.

In more explicit form the conditions to be solved are

$$\frac{\partial U}{\partial t} = \left(1 - \frac{2m}{r}\right)\frac{\partial U}{\partial r}, \qquad \frac{\partial V}{\partial t} = -\left(1 - \frac{2m}{r}\right)\frac{\partial V}{\partial r}$$

2.3 Solving the Conditions: Tortoise Coordinate

To see what to do next, recall what these conditions would look like for null coordinates in Minkowski spacetime:

$$\frac{\partial U}{\partial t} = \frac{\partial U}{\partial r}, \qquad \frac{\partial V}{\partial t} = -\frac{\partial V}{\partial r},$$

which would yield the result that U depends only on $t+r$ and V depends only on $t-r$. To get the conditions into this form, we need a new radial coordinate r^* such that

$$\left(1 - \frac{2m}{r}\right)\frac{\partial}{\partial r} = \frac{\partial}{\partial r^*}$$

From the chain rule for partial derivatives,

$$\frac{\partial}{\partial r} = \frac{\partial r^*}{\partial r}\frac{\partial}{\partial r^*}$$

so we have

$$\left(1 - \frac{2m}{r}\right)\frac{\partial r^*}{\partial r}\frac{\partial}{\partial r^*} = \frac{\partial}{\partial r^*}$$

or

$$\frac{\partial r^*}{\partial r} = \frac{1}{1 - \frac{2m}{r}},$$

which can be re-written as follows

$$\frac{\partial r^*}{\partial r} = \frac{r}{r - 2m} = 1 + \frac{2m}{r - 2m}$$