CS 310 (sec 20) - Spring 2005 - Midterm Exam (solutions)

SOLUTIONS

1. (Logic)

(1) Show that $\neg(p \leftrightarrow q)$ and $(\neg p) \leftrightarrow q$ are logically equivalent.

(2) Determine the truth value of each of these statements if the universe of discourse of each variable consists of all integers.

(a) $\forall n(n^2 \geq 0)$ (b) $\exists n(n^2 = 2)$ (c) $\forall n(n^2 \geq n)$ (d) $\exists n(n^2 < 0)$

(3) Determine the truth value of each of these statements if the universe of discourse of each variable consists of all real numbers.

(a) $\exists x(x^2 = 2)$ (b) $\exists x(x^2 = -1)$ (c) $\forall x(x^2+2 \geq 1)$ (d) $\forall x(x^2 \neq x)$

Solution:

(1) Using a truth table:

p	q	$\neg p$	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	$(\neg p) \leftrightarrow q$
T	T	F	T	F	F
T	F	F	F	T	T
F	T	T	F	T	T
F	F	T	T	F	F

They have the same truth values

(2) (a) TRUE (n^2 is always in fact non-negative).

(b) FALSE ($\sqrt{2}$ is not an integer.)

(c) TRUE (it can be false only for $0 < n < 1$, but there are no integers between 0 and 1.)

(d) FALSE (it is the negation of (a)).

(3) (a) TRUE ($\sqrt{2}$ is a real number.)

(b) FALSE ($i = \sqrt{-1}$ is not real.)

(c) TRUE (since $x^2 \geq 0$).

(d) FALSE (counterexamples: $x = 0$ and $x = 1$.)

2. (Sets)

1. Let $A_i = \{1, 2, 3, \ldots, i\}$ for $i = 1, 2, 3, \ldots$. Find

   (a) $\displaystyle\bigcup_{i=1}^{n} A_i$ (b) $\displaystyle\bigcap_{i=1}^{n} A_i$

2. Let $A_i = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots, i\}$ for $i = 1, 2, 3, \ldots$. Find

   (a) $\displaystyle\bigcup_{i=1}^{n} A_i$ (b) $\displaystyle\bigcap_{i=1}^{n} A_i$

3. Let A_i be the set of all nonempty bit strings (that is, bit strings over $\{0, 1\}$ of length at least one) of length not exceeding i. Find

   (a) $\displaystyle\bigcup_{i=1}^{n} A_i$ (b) $\displaystyle\bigcap_{i=1}^{n} A_i$

Solution:

1. (a) $\displaystyle\bigcup_{i=1}^{n} A_i = A_n = \{1, 2, 3, \ldots, n\}$

   (b) $\displaystyle\bigcap_{i=1}^{n} A_i = A_1 = \{1\}$

2. (a) $\displaystyle\bigcup_{i=1}^{n} A_i = A_n = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots, n\}$

   (b) $\displaystyle\bigcap_{i=1}^{n} A_i = A_1 = \{\ldots, -2, -1, 0, 1\}$

3. (a) $\displaystyle\bigcup_{i=1}^{n} A_i = A_n = \{s \in \{0, 1\}^* \mid 1 \leq |s| \leq n\}$

   (b) $\displaystyle\bigcap_{i=1}^{n} A_i = A_1 = \{0, 1\}$

3. (Relations) For each of the following relations defined on the set of positive integers $\mathbb{Z}^+$, determine which ones are partial orders, and which ones are equivalence relations (some may be both or none).

(a) $x \mathcal{R} y \Leftrightarrow x \leq y$.

(b) $x \mathcal{R} y \Leftrightarrow x < y$.

(c) $x \mathcal{R} y \Leftrightarrow x \not< y$.

(d) $x \mathcal{R} y \Leftrightarrow$ for every prime number p, if p divides x then p divides y.

(e) $x \mathcal{R} y \Leftrightarrow$ for every prime number p, p divides x if and only if p divides y.

(f) $x \mathcal{R} y \Leftrightarrow x = y$.

Solution:

(a) Partial order.

(b) None (it is not reflexive).

(c) Partial order (it is the same as $x \geq y$.)

(d) None. It is not an order relation because it is not antisymmetric (e.g., $2 \mathcal{R} 4$ and $4 \mathcal{R} 2$, but $2 \neq 4$.) It is not an equivalence relation because it is not symmetric (e.g. $2 \mathcal{R} 6$, but 6 is not related to 2.)

(e) Equivalence—x and y are related precisely when they are divided by the same set of primes. It is not a partial order because it is not antisymmetric (e.g., $12 \mathcal{R} 18$ and $18 \mathcal{R} 12$, because both are divided precisely by the primes 2 and 3, but $12 \neq 18$.)

(f) Both, it is an *equivalence* relation (obviously), and it is also a *partial order* relation. Note that it verifies the antisymmetric property, because $x \mathcal{R} y$ and $y \mathcal{R} x$ means $x = y$ and $y = x$, which in fact implies $x = y$.