Suggested Solutions to Midterm Exam
Econ 511b (Part I), Spring 2004

1. **Consider a competitive equilibrium neoclassical growth model populated by identical consumers whose preferences over consumption streams are given by $\sum_{t=0}^{\infty} \beta^t u(c_t)$. Consumers do not value leisure and are endowed with k_0 units of capital in period 0 and with one unit of labor in every period. Consumers rent the services of capital and labor in competitive markets to profit-maximizing firms with identical constantreturns-to-scale production functions. Capital depreciates fully in one period. The government, which balances its budget in every period, taxes capital income at a (time-invariant) proportional rate τ and returns the proceeds to consumers in the form of a lump-sum subsidy to income.**

   (a) **Carefully define a recursive competitive equilibrium for this economy.**

   A recursive competitive equilibrium for the economy with capital income taxation is a set of functions:

$$\begin{aligned} \textit{price function} &: r\left(\overline{k}\right), w\left(\overline{k}\right) \\ \textit{policy function} &: k' = g\left(k, \overline{k}\right) \\ \textit{value function} &: v\left(k, \overline{k}\right) \\ \textit{Taxation} &: T\left(\overline{k}\right) \\ \textit{transition function} &: \overline{k}' = G\left(\overline{k}\right) \end{aligned}$$

   such that:

   (1) Given $r\left(\overline{k}\right), w\left(\overline{k}\right)$, $k' = g\left(k, \overline{k}\right)$ and $v\left(k, \overline{k}\right)$ solves consumer's problem:

$$\begin{aligned} v\left(k, \overline{k}\right) &= \max_{\{c, k'\}} u(c) + \beta v\left(k', \overline{k}'\right) \\ & s.t. \\ & c + k' = (1-\tau) r\left(\overline{k}\right) k + w\left(\overline{k}\right) + T\left(\overline{k}\right) \\ & \overline{k}' = G\left(\overline{k}\right) \end{aligned}$$

   (2) Price is competitively determined:

$$\begin{aligned} r\left(\overline{k}\right) &= F_1\left(\overline{k}, 1\right) \\ w\left(\overline{k}\right) &= F_2\left(\overline{k}, 1\right) \end{aligned}$$

   (3) Government balances its budget

$$T\left(\overline{k}\right) = \tau r\left(\overline{k}\right) \overline{k}$$

   (4) Consistency:

$$G\left(\overline{k}\right) = g\left(\overline{k}, \overline{k}\right)$$

(b) Show that the competitive equilibrium allocation for this economy is identical to the allocation chosen by a social planner whose preferences over consumption streams are given by $\sum_{t=0}^{\infty} \widetilde{\beta}^t u(c_t)$, where $\widetilde{\beta}$ is a "distorted" discount rate that differs from the discount rate of a typical consumer. Express the distorted discount rate in terms of τ and β.

We can solve for F.O.C. as

$$u'(c) = \beta v_1 \left(k', \overline{k}'\right)$$

Envelope condition gives us

$$v_1 \left(k, \overline{k}\right) = u'(c)(1-\tau) r\left(\overline{k}\right)$$

Therefore, we get the Euler equation as

$$\begin{aligned} & u'(c) = \beta(1-\tau) u'(c') r\left(\overline{k}'\right) \\ \Rightarrow \ & u'\left((1-\tau) r\left(\overline{k}\right) k + w\left(\overline{k}\right) + T\left(\overline{k}\right) - \overline{k}'\right) \\ = \ & \beta(1-\tau) u'\left((1-\tau) r\left(\overline{k}'\right) k' + w\left(\overline{k}'\right) + T\left(\overline{k}'\right) - \overline{k}''\right) r\left(\overline{k}'\right) \end{aligned}$$

After plug into price $\left(r\left(\overline{k}\right), w\left(\overline{k}\right)\right)$, taxation $\left(T\left(\overline{k}\right)\right)$, and equilibrium condition $\left(k = \overline{k}\right)$, it becomes

$$\begin{aligned} & u'\left((1-\tau) F_1\left(\overline{k}, 1\right) \overline{k} + F_2\left(\overline{k}, 1\right) + \tau F_1\left(\overline{k}, 1\right) \overline{k} - \overline{k}'\right) \\ = \ & \beta(1-\tau) u'\left((1-\tau) F_1\left(\overline{k}', 1\right) \overline{k}' + F_2\left(\overline{k}', 1\right) + \tau F_1\left(\overline{k}', 1\right) \overline{k}' - \overline{k}''\right) F_1\left(\overline{k}', 1\right) \\ \Rightarrow \ & u'\left(F\left(\overline{k}, 1\right) - \overline{k}'\right) = \beta(1-\tau) u'\left(F\left(\overline{k}', 1\right) - \overline{k}''\right) F_1\left(\overline{k}', 1\right) \end{aligned}$$

Compare this to the Euler equation for central planning problem

$$u'\left(F\left(\overline{k}, 1\right) - \overline{k}'\right) = \widetilde{\beta} u'\left(F\left(\overline{k}', 1\right) - \overline{k}''\right) F_1\left(\overline{k}', 1\right)$$

We have

$$\widetilde{\beta} = \beta(1-\tau)$$

Therefore, the competitive equilibrium allocation for this economy is identical to the allocation chosen by a social planner whose preferences over consumption streams are given by $\sum_{t=0}^{\infty} \widetilde{\beta}^t u(c_t)$, where $\widetilde{\beta} = \beta(1-\tau)$.

(c) Does the result from part (b) continue to hold if consumers value leisure and the government taxes both labor income and capital income at a proportional rate τ? Explain why or why not.

We start from defining an equilibrium with valued leisure.

A recursive competitive equilibrium for the economy with valued leisure is a set of functions:

$$\begin{aligned} price\ function &: r\left(\overline{k}\right), w\left(\overline{k}\right) \\ policy\ function &: k^{'} = g_k\left(k, \overline{k}\right), l = g_n\left(k, \overline{k}\right) \\ value\ function &: v\left(k, \overline{k}\right) \\ Taxation &: T\left(\overline{k}\right) \\ transition\ function &: \overline{k}^{'} = G\left(\overline{k}\right) \end{aligned}$$

such that:

(1) Given $r\left(\overline{k}\right), w\left(\overline{k}\right)$, $k^{'} = g_k\left(k, \overline{k}\right), n = g_n\left(k, \overline{k}\right)$ and $v\left(k, \overline{k}\right)$ solves consumer's problem:

$$\begin{aligned} v\left(k, \overline{k}\right) &= \max_{\left\{c,n,k^{'}\right\}} u\left(c, 1-n\right) + \beta v\left(k^{'}, \overline{k}^{'}\right) \\ & s.t. \\ & c + k^{'} = (1-\tau)\left(r\left(\overline{k}\right)k + w\left(\overline{k}\right)n\right) + T\left(\overline{k}\right) \\ & \overline{k}^{'} = G\left(\overline{k}\right) \end{aligned}$$

(2) Price is competitively determined:

$$\begin{aligned} r\left(\overline{k}\right) &= F_1\left(\overline{k}, g_n\left(\overline{k}, \overline{k}\right)\right) \\ w\left(\overline{k}\right) &= F_2\left(\overline{k}, g_n\left(\overline{k}, \overline{k}\right)\right) \end{aligned}$$

(3) Government balances its budget

$$T\left(\overline{k}\right) = \tau\left(r\left(\overline{k}\right)\overline{k} + w\left(\overline{k}\right)g_n\left(\overline{k}, \overline{k}\right)\right)$$

(4) Consistency:

$$G\left(\overline{k}\right) = g\left(\overline{k}, \overline{k}\right)$$

Solve for this, we can get the F.O.C. as

$$\begin{aligned} \{k'\} &: u_1(c, 1-n) = \beta(1-\tau)u_1(c', 1-n')r\left(\overline{k}^{'}\right) \\ \{n\} &: u_1(c, 1-n)(1-\tau)w\left(\overline{k}\right) = u_2(c, 1-n) \end{aligned}$$

After plug into c, price $(r\left(\overline{k}\right), w\left(\overline{k}\right))$, taxation $(T\left(\overline{k}\right))$, and equilibrium condition $(k = \overline{k})$, the F.O.C. becomes

$$\begin{aligned} \{k'\} &: u_1\left(F\left(\overline{k}, n\right) - \overline{k}^{'}, 1-n\right) = \beta(1-\tau)u_1\left(F\left(\overline{k}^{'}, n'\right) - \overline{k}^{''}, 1-n'\right)F_1\left(\overline{k}^{'}, n'\right) \\ \{n\} &: (1-\tau)u_1\left(F\left(\overline{k}, n\right) - \overline{k}^{'}, 1-n\right)F_2\left(\overline{k}, n\right) = u_2\left(F\left(\overline{k}, n\right) - \overline{k}^{'}, 1-n\right) \end{aligned}$$