MATH 315-02 Review for Midterm Exam / March 2002 [1]

1st Order Linear Equations

- **Linear equations**

The standard form of a first order linear equation is $y' + p(t)y = g(t)$. Use the *integrating factor* $\mu(t) = e^{\int p(t)\,dt}$ to transform the ODE into the form $(\mu y)' = \mu(t)g(t)$. This has general solution

$$y = \frac{\int \mu(t)g(t)\,dt + C}{\mu(t)}.$$

- **Separable equations**

A first order ODE is *separable* if it can be put into the form $M(x)dx + N(y)dy = 0$. Then the ODE can be solved implicitly by integration: $\int M(x)\,dx + \int N(y)\,dy = 0$.

- **Some theory for 1st order equations**

Theorem. *If p and g are continuous on the interval $a < t < b$ and t_0 is inside this interval, then for every real number y_0, the IVP*

$$y' + p(t)y = g(t), \quad y(t_0) = y_0$$

has a unique solution valid on the interval.

Theorem. *If f and f_y are continuous on a rectangle $a < t < b$, $c < y < d$ and the point (t_0, y_0) is inside the rectangle, then the IVP*

$$y' = f(t, y), \quad y(t_0) = y_0$$

has a unique solution valid on some interval $t_0 - h < t < t_0 + h$.

- **Exact equations**

An equation in the form $M(x,y)dx + N(x,y)dy = 0$ is *exact* if $M_y = N_x$. Then find a function $\Psi(x,y)$ such that $\Psi_x = M$ and $\Psi_y = N$. For example, you could set $\Psi(x,y) = \int M(x,y)\,dx + h(y)$. Then set $\Psi_y = \frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right) + h'(y) = N$ and solve for $h(y)$. Since $\frac{d}{dx}\Psi = M(x,y) + N(x,y)y' = 0$, the general solution is $\Psi(x,y) = C$.

- **Numerical approximation: Euler's method**

Given an IVP $y' = f(t,y)$, $y(t_0) = y_0$, approximate the solution at $t = a$ via Euler's method:
With step size $h = (a - t_0)/n$, compute:

$$t_k = t_0 + hk, \qquad y_k = y_{k-1} + hf(t_{k-1}, y_{k-1}), \qquad k = 1, 2 \ldots, n.$$

Then y_n is an approximation of $y(a)$.

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[1] Thanks to Prof. David Boyd for providing his review notes on which this document is based.

2nd Order Linear Equations

A differential equation is *second order* and *linear* if it can be put into the form

$$P(t)y'' + Q(t)y' + R(t)y = G(t).$$

A second order ODE is in standard form if it has the form

$$y'' + p(t)y' + q(t)y = g(t).$$

The equation is called *homogeneous* if $g(t) \equiv 0$. Otherwise, the equation is *non-homogeneous*.

Theorem. *Given an IVP*

$$\begin{gathered} y'' + p(t)y' + q(t)y = g(t) \\ y(t_0) = y_0, \quad y'(t_0) = y_0', \end{gathered}$$

if p, q and g are continuous on an open interval I containing t_0, then the IVP has a unique solution which is valid on I.

- **Homogeneous Equations**

If y_1 and y_2 are solutions to a homogeneous equation $y'' + p(t)y' + q(t)y = 0$, then any function of the form $y = c_1y_1 + c_2y_2$ is also a solution where c_1 and c_2 are any constants.

We call $\{y_1, y_2\}$ a *fundamental set of solutions* if *every* solution to the IVP

$$\begin{gathered} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = y_0, \quad y'(t_0) = y_0', \end{gathered}$$

has the form $y = c_1y_1 + c_2y_2$. In this case, c_1 and c_2 are found by solving

$$\begin{gathered} c_1y_1(t_0) + c_2y_2(t_0) = y_0 \\ c_1y_1'(t_0) + c_2y_2'(t_0) = y_0' \,. \end{gathered}$$

This can always be done if the *Wronskian*

$$W(y_1, y_2)(t_0) := \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0.$$

- **Homogeneous equations with constant coefficients**

Start with an ODE

$$ay'' + by' + cy = 0. \tag{1}$$

where a, b, and c are constants, $a \neq 0$.

The *characteristic equation* of (1) is the polynomial equation

$$ar^2 + br + c = 0. \tag{2}$$

If the roots of (2) are distinct real numbers r_1 and r_2, then the general solution to (1) is

$$y = c_1e^{r_1t} + c_2e^{r_2t}.$$

If there is only one real root r_1 of (2) (a double root), then the general solution to (1) is

$$y = c_1e^{r_1t} + c_2te^{r_1t}.$$

If (2) has complex roots $\lambda \pm i\mu$, $(\mu \neq 0)$, then the general solution to (1) is

$$y = c_1e^{\lambda t}\cos(\mu t) + c_2e^{\lambda t}\sin(\mu t).$$

- **Non-homogeneous equations**

If Y is any solution to

(3) $$y'' + p(t)y' + q(t)y = g(t)$$

and $c_1y_1 + c_2y_2$ is the general solution for

$$y'' + p(t)y' + q(t)y = 0,$$

then the general solution to (3) is

$$y = c_1y_1 + c_2y_2 + Y.$$

(Terminology: $y_c := c_1y_1 + c_2y_2$ is the *complementary* solution and Y is called a *particular solution.*)

- **Method of undetermined coefficients**

The summary for second and higher order equations is given later in this review.

- **Variation of parameters**

If

$$y'' + p(t)y' + q(t)y = 0$$

has fundamental set of solutions $\{y_1, y_2\}$, then

$$y'' + p(t)y' + q(t)y = g(t)$$

has general solution $y = c_1y_1(t) + c_2y_2(t) + Y(t)$, where

$$Y(t) = -y_1(t)\int \frac{y_2(t)\,g(t)}{W(y_1,y_2)(t)}\,dt + y_2(t)\int \frac{y_1(t)\,g(t)}{W(y_1,y_2)(t)}\,dt.$$

- **Mechanical vibrations**

Model for motion of a vibrating spring/mass system: If $u(t)$ is the displacement of the mass from *equilibrium position* (positive direction is downward) then

$$mu'' + \gamma u' + ku = F(t),$$

where m is the mass, γ is the *damping constant*, k is the *spring constant* and $F(t)$ is the sum of any additional forces acting on the spring.

The *damping force* is proportional to the speed, so

$$\text{damping force at time } t \;=\; \gamma\,(\text{speed at time } t).$$

If the relaxed spring of length l is stretched an additional distance L (to the equilibrium position) by a mass m, then the spring constant is

$$k = \frac{mg}{L}.$$

Undamped free vibrations: $F(t) \equiv 0$, $\gamma = 0$. Then ODE is $mu'' + ku = 0$, which has general solution

$$u = A\cos(\omega_o t) + B\sin(\omega_0 t),$$

where $\omega_0 = \sqrt{k/m}$. If $R = \sqrt{A^2 + B^2}$ and $\tan\delta = B/A$, $A = R\cos\delta$, $B = R\sin\delta$, then

$$u = R\cos(\omega_0 t - \delta).$$

R is the *amplitude*, ω_0 is the *frequency*, $2\pi/\omega_0$ is the *period* and δ is the *phase*.