# Probability Laws

The purpose of this document is to review and illustrate the laws of probability.
The most important of these laws are the following.

The Addition Rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and for mutually exclusive events $P(A \cup B \cup C) = P(A) + P(B) + P(C)$.
Definition of Mutually exclusive: $P(A \cap B) = 0$
The Probability of a Complement: $P(\overline{A}) = 1 - P(A)$
The Multiplication Rule: $P(A \cap B) = P(A|B)P(B)$ or $P(A|B) = \frac{P(A \cap B)}{P(B)}$
The definition of independence: $A$ and $B$ are independent if $P(A|B) = P(A)$. **Only if this is true can we assume that** $P(A \cap B) = P(A) \cdot P(B)$. In particular, two events cannot be both mutually exclusive and independent, except under unlikely circumstances.

The problem is due to Bassett et al* and says that on a given Sunday the probability that a fisherman goes to the sea is .50; the probability that he goes to a river is .25 and the probability that he goes to a lake is also .25. If he goes to the sea, the probability that he catches a fish is .80; if he goes to the river, the probability that he catches a fish is .40; if he goes to a lake, the probability that he catches a fish is .60.
**What the problem says:** The most important thing to note here is that the last three probabilities are <u>conditional</u> probabilities, not joint probabilities. They are conditional probabilities, not joint probabilities, because they talk about catching a fish if something else happens. So let us say $S$ is the event that he goes to the sea; $R$ is the event that he goes to the river; $L$ is the event that he goes to the lake and $F$ is the event that he catches a fish.

The first three probabilities are the probabilities of mutually exclusive events. They are $P(S) = .5$, $P(R) = .25$ and $P(L) = .25$. Since these add to 1, it is implied that it is absolutely certain that he will go fishing on Sunday and it is also implied that he can go only one place on a given Sunday.

The second three probabilities are the conditional probabilities of events that can only occur in a specific location. They are $P(F|S) = .80$, $P(F|R) = .40$ and $P(F|L) = .60$. If we wish, we can use the multiplication rule to construct a joint probability table.

But there is an implied event of which we have said nothing. This is the event $\overline{F}$, that he does not catch a fish. Just as we do not know $P(F)$ yet, we also do not know $P(\overline{F})$ but we can find out by construction a joint probability table. It may be useful to use the rule for constructing the probability of a complement to get the probabilities of not catching a fish in each of the three locations. $P(\overline{F}|S) = 1 - P(F|S) = 1 - .80 = .20$, $P(\overline{F}|R) = 1 - P(F|R) = 1 - .40 = .60$ and $P(\overline{F}|L) = 1 - P(F|L) = 1 - .60 = .40$. On the other hand we could get these from the joint probabilities after we start constructing the table.

So we can start by finding the joint probabilities using the multiplication rule.
$P(F \cap S) = P(F|S)P(S) = .80(.5) = .40$, $P(F \cap R) = P(F|R)P(R) = .40(.25) = .10$ and $P(F \cap L) = P(F|L)P(L) = .60(.25) = .15$. Now we have all we need to construct the table.

| | $S$ | $R$ | $L$ | $total$ |
|---|---|---|---|---|
| $F$ | .40 | .10 | .15 | |
| $\overline{F}$ | | | | |
| $total$ | .50 | .25 | .25 | 1.00 |

. If we add across using the addition rule and realizing that the three joint events on the top line are mutually exclusive , we can find what is probably the most important probability, the probability that he catches a fish, which is .65. If we want to do this formally, we can write $P(F) = P(F \cap S) + P(F \cap R) + P(F \cap L) = .40 + .10 + .15 = .65$. We can also finish the table

*Bassett, Bremner, Morgan, Jolliffe, Jones and North. <u>Statistics: Problems and Solutions,</u> 2nd ed., World Scientific, Singapore, 2000.

by realizing that it must add up. Our complete table is thus

| | $S$ | $R$ | $L$ | $total$ |
|---|---|---|---|---|
| $F$ | .40 | .10 | .15 | .65 |
| $\bar{F}$ | .10 | .15 | .10 | .35 |
| $total$ | .50 | .25 | .25 | 1.00 |

. We can now get our conditional probabilities of not catching a fish a second way using the multiplication rule. $P(\bar{F}|S) = \frac{P(\bar{F} \cap S)}{P(S)} = \frac{.10}{.50} = .20$ etc. But we could also have used the conditional probabilities of not catching a fish that we got earlier to get the second line on the table by saying $P(\bar{F} \cap S) = P(\bar{F}|S)P(S) = .20(.5) = .10$ etc.

Of course, the table shows that none of the events are independent since we cannot get any probabilities inside the table by multiplying probabilities outside the table. For instance $.40 = P(F \cap S) \neq (P(F))(P(S)) = .50(.65) = .325$. Though much of this problem, which deals wit probabilities of catching a fish on one or more of several consecutive Sundays or of two similar fishermen meeting on two consecutive Sundays, we can at least look at the following.

a) The probability of catching a fish on two successive Sundays. If we assume that the probabilities stay the same which means i) the fisherman does not learn by experience and ii) if we catch a fish this week it does not significantly deplete the stock of fish available on the next Sunday we can assume the events $F_1$, a fish on the first Sunday and $F_2$, the probability of a fish on the second Sunday are independent so that $P(F_1 \cap F_2) = P(F_1)(P(F_2)) = .65(.65) = .4225$. This is an example of the Binomial distribution, which will give us a formula for finding the probability of $x$ successes in $n$ tries when i) the results of any one try are not influenced by the outcome of another try and ii) the probability of success on any one try stays constant.

b) We can also use Bayes' rule to answer the question 'if the fisherman comes home on a given Sunday without a fish, what is the probability that he has been to each of the three locations? The first of these is a conditional probability $P(S|\bar{F})$ and we can find it by using

$P(S|\bar{F}) = \frac{P(\bar{F}|S)P(S)}{P(\bar{F})} = \frac{.20(.5)}{.35} = \frac{.10}{.35} = .2857$ or we can simple say, using the multiplication rule and our table that $P(S|\bar{F}) = \frac{P(S \cap \bar{F})}{P(\bar{F})} = \frac{.10}{.35} = .2857$. Of course if this was the question given at the beginning of the problem, we would have had to construct all three of the probabilities of not catching a fish, used them to get the joint probabilities in the second row of the table and then added them together to get the total probability of not catching a fish. You might want to try the rest of these for practice. The authors say that $P(R|\bar{F}) = .4286$ and $P(L|\bar{F}) = .2857$. Note that, since our assumptions lead us to the conclusion that if he didn't catch a fish he must have been somewhere, the three conditional probabilities add to one.

*Bassett, Bremner, Morgan, Jolliffe, Jones and North. Statistics: Problems and Solutions, 2nd ed., World Scientific, Singapore, 2000.