6/2

midterm 6/10

$a\dot{x} + bx = u$

solve $x(t) = ?$

$$Y = Y_h + Y_p$$

Y_h → homogenous response (effect of initial conditions)

Y_p → particular response (effect of input)

$G(s) = \frac{Y(s)}{U(s)} \rightarrow Y(s) = U(s)G(s)$

$$Y(t) = L^{-1}[Y(s)] = L^{-1}[U(s)G(s)]$$

$$F(s) = \int_0^{\infty} e^{-st} f(t) = L(f(t))$$

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}F(s)ds = L^{-1}(F(s))$$

→ use Laplace tables to solve

$$L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$$

$$L^{-1}\left(\frac{1}{s(s+a)}\right) = \frac{1}{a}(1-e^{-at})$$

$$L^{-1}\left(\frac{1}{(s+a)(s+b)}\right) = \frac{1}{b-a}(e^{-at} - e^{-bt})$$

$$L^{-1}\left(\frac{\omega_n^2}{s^2+2\eta\omega_n s+\omega_n^2}\right) = \frac{\omega_n}{\sqrt{1-\eta^2}} e^{-\eta\omega_n t}\sin(\omega_n t\sqrt{1-\eta^2}) \quad 0<\eta<1$$

$G(s) = \frac{10}{s+1}$ $\quad u(t) = \delta(t)$ $\quad \mathcal{L}[\delta(t)] = 1$ $\quad u(t) = t$ $\quad \mathcal{L}[t] = \frac{1}{s^2}$

$Y(s) = \frac{10(1)}{s+1} \rightarrow \mathcal{L}^{-1}\left[\frac{10}{s+1}\right] = 10e^{-t}$

$Y(s) = \left[\frac{1}{s^2}\right]\left[\frac{10}{s+1}\right] \rightarrow \mathcal{L}^{-1}\left[\frac{10}{s^2+1}\right] = 10\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)}\right] = 10(t-1+e^{-t})$

$$G(s) = \frac{300}{s^3+36s} \qquad U(s) = \frac{1}{s}$$

poles : $0, \pm 6j$

$$Y(s) = \frac{1}{s}\,\frac{300}{s^3+36s} \rightarrow \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{300}{s^4+36s^2}\right] = 300\,\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+36)}\right]$$

$$\mathcal{L}^{-1}\left[\frac{\omega^3}{s^2(s^2+\omega^2)}\right] = \omega t - \sin \omega t$$

$$y(t) = \frac{300}{216}\,\mathcal{L}^{-1}\left[\frac{216}{s^2(s^2+36)}\right] = \frac{300}{216}(6t - \sin 6t)$$

$$x(t) = x_h(t) + x_p(t) \qquad u(t) = 1 \qquad T\frac{dx}{dt} + x = u$$

$$y = x$$

homogeneous response

$$u = 0$$

$$x(0) = x_0 \qquad T\frac{dx}{dt} + x = 0 \longrightarrow \dot{x} = -\frac{1}{T}x$$

$$x_h(t) = ce^{\lambda t}$$

solving characteristic equation

$$\dot{x}_h = \lambda ce^{\lambda t} \qquad \lambda ce^{\lambda t} = -\frac{1}{T}ce^{\lambda t} \rightarrow \lambda = -\frac{1}{T}$$

$$x_h(t) = ce^{-\frac{t}{T}} \longrightarrow x(0) = c = x_0$$

$$x_h(t) = x_0 e^{-\frac{t}{T}}$$

particular response

$$T\frac{dx}{dt} + x = 1$$

$$x_p = c(t)c^{\lambda t} \rightarrow c(t)e^{-\frac{t}{T}}$$

$$T\dot{x}_p + x_p = 1 \qquad \dot{x}_p = \dot{c}(t)e^{-\frac{t}{T}} + c(t)\left(-\frac{1}{T}e^{-\frac{t}{T}}\right)$$

$$T\left(\dot{c}(t)e^{-\frac{t}{T}} + c(t)\left(-\frac{1}{T}e^{-\frac{t}{T}}\right)\right) + c(t)e^{-\frac{t}{T}} = 1$$

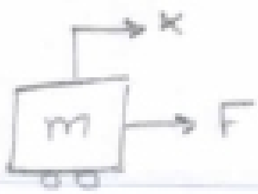
$$T\dot{c}(t)e^{-\frac{t}{T}} = 1 \longrightarrow \dot{c}(t) = \frac{1}{T}e^{\frac{t}{T}} \longrightarrow c(t) = \int_0^t \frac{1}{T}e^{t/T}\,dt \rightarrow c(t) = e^{t/T} - 1$$

$$x_p = 1 - e^{-t/T}$$

complete Response

$$x = x_h + x_p$$

$$x = x_0 e^{-\frac{t}{T}} + 1 - e^{-\frac{t}{T}} \rightarrow x = 1 + e^{-\frac{t}{T}}(x_0 - 1)$$



$m\ddot{x} = F$

$\underline{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$

input = F
output = x (position)

$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \qquad y = [1 \; 0]\, \underline{x}$

$y_h = ?$ if $x(0) = x_0$

$x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$ → initial position, → initial velocity

$y_h = C e^{At} x_0$

$At = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \qquad (At)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$e^{At} = I + At + \frac{1}{2!}(At)^2 + \dots$

$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} + \cancel{\frac{1}{2!}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} \to 0$

$e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

$y_h = [1 \; 0] \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = [1 \;\; t] \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = x_{10} + t x_{20}$

<u>particular response</u>

$x_p = e^{At} c(t)$

$e^{At} \dot{c}(t) = Bu(t) \rightarrow \dot{c} = e^{-At} Bu(t) \rightarrow c = \int e^{-At} Bu(T)\, dT$

$x_p = \int_0^t e^{A(t-T)} Bu(T)\, dT$

$x = x_h + x_p = e^{At} x(0) + \int_0^t e^{A(t-T)} Bu(T)\, dT$

$y(t) = Cx(t) + Du(t)$

$y(t) = \underbrace{Ce^{At} x(0)}_{\hookrightarrow\, x_{1,0} + x_{2,0}} + \underbrace{\int_0^t Ce^{A(t-T)} Bu(T)\, dT}\, + Du(T)$

$\hookrightarrow Ce^{A(t-T)} Bu(T) = [1 \; 0] \begin{bmatrix} 1 & t-T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} (1) = [1 \;\; t-T] \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} = \frac{t}{m} - \frac{T}{m}$

$\rightarrow \int_0^t \frac{t}{m} - \frac{T}{m}\, dT = \frac{tT}{m} - \frac{T^2}{2m} \Big|_0^t = \frac{t^2}{m} - \frac{t^2}{2m} = \frac{t^2}{2m}$

$y = y_p + y_h = x_p + t x_{20} + \frac{t^2}{2m}$