

Chapter 2

System Modeling

... I asked Fermi whether he was not impressed by the agreement between our calculated numbers and his measured numbers. He replied, "How many arbitrary parameters did you use for your calculations?" I thought for a moment about our cut-off procedures and said, "Four." He said, "I remember my friend Johnny von Neumann used to say, with four parameters I can fit an elephant, and with five I can make him wiggle his trunk."

Freeman Dyson on describing the predictions of his model for meson-proton scattering to Enrico Fermi in 1953 [8].

A model is a precise representation of a system's dynamics used to answer questions via analysis and simulation. The model we choose depends on the questions that we wish to answer, and so there may be multiple models for a single physical system, with different levels of fidelity depending on the phenomena of interest. In this chapter we provide an introduction to the concept of modeling, and provide some basic material on two specific methods that are commonly used in feedback and control systems: differential equations and difference equations.

2.1 Modeling Concepts

A model is a mathematical representation of a physical, biological or information system. Models allow us to reason about a system and make predictions about how a system will behave. In this text, we will mainly be interested in models describing the input/output behavior of systems and often in so-called "state space" form.

Roughly speaking, a dynamical system is one in which the effects of actions do not occur immediately. For example, the velocity of a car does not

change immediately when the gas pedal is pushed nor does the temperature in a room rise instantaneously when an air conditioner is switched on. Similarly, a headache does not vanish right after an aspirin is taken, requiring time to take effect. In business systems, increased funding for a development project does not increase revenues in the short term, although it may do so in the long term (if it was a good investment). All of these are examples of dynamical systems, in which the behavior of the system evolves with time.

Dynamical systems can be viewed in two different ways: the internal view or the external view. The internal view attempts to describe the internal workings of the system and originates from classical mechanics. The prototype problem was describing the motion of the planets. For this problem it was natural to give a complete characterization of the motion of all planets. This involves careful analysis of the effects of gravitational pull and the relative positions of the planets in a system.

The other view on dynamics originated in electrical engineering. The prototype problem was to describe electronic amplifiers. It was natural to view an amplifier as a device that transforms input voltages to output voltages and disregard the internal detail of the amplifier. This resulted in the input/output view of systems. The two different views have been amalgamated in control theory. Models based on the internal view are called internal descriptions, state models, or white box models. The external view is associated with names such as external descriptions, input/output models or black box models. In this book we will mostly use the words state models and input/output models.

In the remainder of this section we provide an overview of some of the key concepts in modeling. The mathematical details introduced here are explored more fully in the remainder of the chapter.

The Heritage of Mechanics

The study of dynamics originated in the attempts to describe planetary motion. The basis was detailed observations of the planets by Tycho Brahe and the results of Kepler who found empirically that the orbits of the planets could be well described by ellipses. Newton embarked on an ambitious program to try to explain why the planets move in ellipses and he found that the motion could be explained by his law of gravitation and the formula that force equals mass times acceleration. In the process he also invented calculus and differential equations. Newton's result was the first example of the idea of reductionism, i.e. that seemingly complicated natural phenomena can be explained by simple physical laws. This became the paradigm of natural

science for many centuries.

One of the triumphs of Newton's mechanics was the observation that the motion of the planets could be predicted based on the current positions and velocities of all planets. It was not necessary to know the past motion. The *state* of a dynamical system is a collection of variables that characterize the motion of a system completely for the purpose of predicting future motion. For a system of planets the state is simply the positions and the velocities of the planets. We call the set of all possible states the *state space*.

A common class of mathematical models for dynamical systems is ordinary differential equations (ODEs). Mathematically, an ODE is written as

$$\frac{dx}{dt} = f(x). \quad (2.1)$$

Here $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ is a vector of real numbers that describes the current state of the system and equation (2.1) describes the rate of change of the state as a function of the state itself. Note that we do not bother to write the vector x any differently than a scalar variable. It will generally be clear from context whether a variable is a vector or scalar quantity.

An example of an ordinary differential equation is the van der Pol equation,

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 - x_1^3 - x_2 \\ \frac{dx_2}{dt} &= x_1, \end{aligned} \quad (2.2)$$

which is a model of an electronic oscillator. The state of the system is represented by two real numbers, x_1 and x_2 . The model (2.2) gives the velocity of the state vector for each value of the state.

The evolution of the states can be described using either a time plot or a phase plot, both of which are shown in Figure 2.1. The time plot, on the left, shows the values of the individual states as a function of time. The phase plot, on the right, shows the *vector field* for the system, which gives the state velocity (represented as an arrow) at every point in the state space. In addition, we have superimposed the traces of some of the states from different conditions. The phase plot gives a strong intuitive representation of the equation as a vector field or a flow. While systems of second order (two states) can be represented in this way, it is unfortunately difficult to visualize equations of higher order using this approach.

The ideas of dynamics and state have had a profound influence on philosophy where they inspired the idea of predestination. If the state of a natural system is known at some time, its future development is completely