MATRIX PRODUCT — **Math 21b, O. Knill**

HOMEWORK: Section 2.4: 4,14,28,40,76,48*,68*

MATRIX PRODUCT. If B is a $p \times m$ matrix and A is a $m \times n$ matrix, then BA is defined as the $p \times n$ matrix with entries $(BA)_{ij} = \sum_{k=1}^{m} B_{ik} A_{kj}$.

EXAMPLE. If B is a 3×4 matrix, and A is a 4×2 matrix then BA is a 3×2 matrix.

$$B = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 1 & 8 & 1 \\ 1 & 0 & 9 & 2 \end{bmatrix},\ A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},\ BA = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 1 & 8 & 1 \\ 1 & 0 & 9 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 13 \\ 14 & 11 \\ 10 & 5 \end{bmatrix}.$$

COMPOSING LINEAR TRANSFORMATIONS. If $S : \mathbb{R}^n \to \mathbb{R}^m, x \mapsto Ax$ and $T : \mathbb{R}^m \to \mathbb{R}^p, y \mapsto By$ are linear transformations, then their composition $T \circ S : x \mapsto B(A(x))$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^p$. The corresponding matrix is the matrix product $B \cdot A$.

EXAMPLE. Find the matrix which is a composition of a rotation around the x-axes by an agle $\pi/2$ followed by a rotation around the z-axes by an angle $\pi/2$.
SOLUTION. The first transformation has the property that $e_1 \to e_1, e_2 \to e_3, e_3 \to -e_2$, the second $e_1 \to e_2, e_2 \to -e_1, e_3 \to e_3$. If A is the matrix belonging to the first transformation and B the second, then BA is the matrix to the composition. The composition maps $e_1 \to -e_2 \to e_3 \to e_1$ is a rotation around a long diagonal. $B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

EXAMPLE. A rotation dilation is the composition of a rotation by $\alpha = \arctan(b/a)$ and a dilation (=scale) by $r = \sqrt{a^2 + b^2}$.

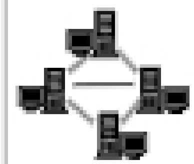
REMARK. Matrix multiplication is a generalization of usual multiplication of numbers or the dot product.

MATRIX ALGEBRA. Note that $AB \neq BA$ in general! Otherwise, the same rules apply as for numbers:
$A(BC) = (AB)C$, $AA^{-1} = A^{-1}A = 1_n$, $(AB)^{-1} = B^{-1}A^{-1}$, $A(B + C) = AB + AC$, $(B + C)A = BA + CA$ etc.

PARTITIONED MATRICES. The entries of matrices can themselves be matrices. If B is a $m \times n$ matrix and A is a $n \times p$ matrix, and assume the entries are $k \times k$ matrices, then BA is a $m \times p$ matrix where each entry $(BA)_{ij} = \sum_{k=1}^{n} B_{ik} A_{kj}$ is a $k \times k$ matrix. Partitioning matrices can be useful to improve the speed of matrix multiplication (i.e. Strassen algorithm).

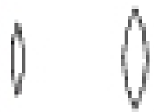
EXAMPLE. If $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, where A_{ij} are $k \times k$ matrices with the property that A_{11} and A_{22} are invertible, then $B = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1} A_{12} A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$ is the inverse of A.

APPLICATIONS. (The material which follows is for motivation puposes only, more applications appear in the homework).



FALSE COLORS. Any color can be represented as a vector (r,g,b), where $r \in [0,1]$ is the red $g \in [0,1]$ is the green and $b \in [0,1]$ is the blue component. Changing colors in a picture means applying a transformation on the cube. Let $T : (r,g,b) \mapsto (g,b,r)$ and $S : (r,g,b) \mapsto (r,g,0)$. What is the composition of these two linear maps?

OPTICS. Matrices help to calculate the motion of light rays through lenses. A light ray $y(s) = x + ms$ in the plane is described by a vector (x,m). Following the light ray over a distance of length L corresponds to the map $(x,m) \mapsto (x+mL, m)$. In the lens, the ray is bent depending on the height x. The transformation in the lens is $(x,m) \mapsto (x, m-kx)$, where k is the strength of the lens.



$$\begin{bmatrix} x \\ m \end{bmatrix} \mapsto A_L \begin{bmatrix} x \\ m \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}, \quad \begin{bmatrix} x \\ m \end{bmatrix} \mapsto B_k \begin{bmatrix} x \\ m \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}.$$

Examples:
1) Eye looking far: $A_R B_k$. 2) Eye looking at distance L: $A_R B_k A_L$.
3) Telescope: $B_{k_2} A_L B_{k_1}$. (More about it in problem 80 in section 2.4).