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- exam next week friday
  - similar to quizes

- moving frame:

Tangent	T	$T \times N = B$	• if you mix up
Normal	N	$N \times B = T$	any of these
binormal	B	$B \times T = N$	you get a negative

position : $r(t)$
velocity : $v(t) = r'(t)$
acceleration : $a(t) = v'(t) = r''(t)$
Jerk : $a'(t) = v''(t) = r'''(t)$

Speed: $|v(t)|$
arc length: $s(t) = \int_0^t |v(u)|\,du$
curvature κ
torsion τ $\quad T = \frac{v(t)}{|v(t)|}$

$$\frac{dT}{ds} = \kappa N \qquad B = T \times N$$

$$\frac{d}{ds}(P) = \frac{d}{dt}(P)\frac{dt}{ds} = \frac{\frac{dP}{dt}}{\frac{ds}{dt}} = \frac{1}{|v(t)|} \cdot \frac{dP}{dt}$$

$$\hookrightarrow \frac{d}{ds}(P) = \frac{d}{dt}(P)\frac{1}{|v(t)|}$$

$$\frac{dB}{ds} = \frac{d}{ds}(T \times N)$$

• anything crossitself is zero

$$= \frac{dT}{ds} \times N + T \times \frac{dN}{ds}$$

$$= \kappa N \times N + T \times \frac{dN}{ds}$$

$$= T \times \frac{dN}{ds}$$ ← • perpendicular to T, also perpendicular to B

$$\frac{dB}{ds} = \underline{-\tau} N \qquad \frac{dN}{ds} = ? \qquad \frac{dN}{ds} = \tau B - \kappa N$$

$$\frac{dN}{ds} = \frac{d}{ds}(B \times T) = \frac{dB}{ds} \times T + B \times \frac{dT}{ds}$$

$$= -\tau N \times T + B \times \kappa N \qquad = \tau B - \kappa N$$

- *ex:* Given $r(t)$
  find T, N, B, κ, τ

$v(t) = |v(t)|T$

$a(t) = \frac{d}{dt}|v(t)|T + |v(t)|\frac{dT}{dt}$

$= \frac{d}{dt}|v(t)|T + |v(t)|\frac{dT}{ds}\frac{ds}{dt}$

$= \frac{d}{dt}|v(t)|T + |v(t)|^2 \kappa N$

$v \times a = |v(t)|T \times \left(\frac{d}{dt}|v(t)|T + |v(t)|^2 \kappa N\right)$

- first part zero because $T \times T$

$= 0 + |v(t)|^3 \kappa B$

$B = \frac{v \times a}{|v \times a|}$

$a'(t) = (\dots)T + (\dots)N + (|v|^3 \kappa \tau)B$

$(v \cdot a)a' = |v|^3 \kappa |v|^3 \kappa \tau$

$\kappa = \frac{|v \times a|}{v^3}$

side work

$|v|^2 \kappa \frac{dN}{dt} = |v|^2 \kappa \frac{dN}{ds}|v|$

$= |v|^3 \kappa(\tau B - \kappa N$

$$\frac{(v \times a) \cdot a'}{|v \times a|^2} = \tau$$

- *ex!* Ellipse
  $r(t) = \langle \cos t, \sin t, 1 - \cos \rangle$
  $v(t) = \langle -\sin t, \cos t, \sin t \rangle$
  $a(t) = \langle -\cos t, -\sin t, \cos t \rangle$
  $a'(t) = \langle \sin t, -\cos t, -\sin t \rangle$

$$v \times a = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin t & \cos t & \sin t \\ -\cos t & -\sin t & \cos t \end{vmatrix} = \hat{i}(1) - \hat{j}(0) + \hat{k}(1) = \langle 1, 0, 1 \rangle$$

$T = \frac{v}{|v|}$

$B = \frac{v \times a}{|v \times a|}$ $\quad \kappa = \frac{|v \times a|}{|v|^3}$ $\quad \frac{(v \times a) \cdot a'}{|v \times a|^2} = \tau$

$N = B \times T$

$(v \times a) \cdot a' = 0$

$|v| = \sqrt{\sin^2 t + \cos^2 t + \sin^2 t} = \sqrt{1 + \sin^2 t}$

$|v \times a| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$

$\boxed{\kappa = \frac{\sqrt{2}}{\sqrt{1+\sin^2 t}^3}}$ $\quad \boxed{\tau = 0}$

$\boxed{T = \frac{1}{\sqrt{1+\sin^2 t}} \langle -\sin t, \cos t, \sin t \rangle}$

$\boxed{B = \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle}$

$N = B \times T = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ -\sin t & \cos t & \sin t \end{vmatrix} \frac{1}{\sqrt{2}\sqrt{1+\sin^2 t}}$

$\left[\hat{i}(-\cos t) - \hat{j}(2\sin t) + \hat{k}(\cos t) \right] \left(\frac{1}{\sqrt{2}\sqrt{1+\sin^2 t}} \right)$

$\boxed{N = \langle -\cos t, -2\sin t, \cos t \rangle \frac{1}{\sqrt{2}\sqrt{1+2\sin^2 t}}}$

- instead of doing cross product you can do this:

$a = \frac{d}{dt}|v| T + |v|^2 \kappa N$

$|v|^2 \kappa N = a - \frac{d}{dt}|v| T$ $\qquad N = \frac{a - \frac{d|v|}{dt} T}{\left| a - \frac{d|v|}{dt} T \right|}$