9/2/14

Arc length

position function $r(t)$

s: length

$\Delta s_i \approx |v(t_i)|\,\Delta t$

$$\text{length} = \sum \Delta s_i$$
$$= \sum |v(t_i)|\,\Delta t_i$$
$$\int_0^{a} |v(t_i)|\,dt$$

$$\text{length} = \int_a^b |v(t)|\,dt$$
$r(t)$ for $a \leq t \leq b$

Arc length parameter

$$s(t) = \int_a^t |v(t)|\,dt \Rightarrow s(t) = \int_a^t |v(\tau)|\,d\tau$$

EX) $r_1 = \langle 3+t^3,\ 6-2t^3,\ 1+4t^3 \rangle$ $\quad 0 \leq t \leq 1$ $\quad (r_1 \Rightarrow L = \text{length})$

velocity $= r'(t) = \langle 3t^2, -6t^2, 12t^2 \rangle$
$$= 3t^2 \langle 1, -2, 4 \rangle$$
$$|v(t)| = |3t^2| \sqrt{1^2+(-2)^2+4^2}$$
$$= 3t^2\sqrt{21}$$

Arc length: $s(t) = \int_0^t 3\tau^2\sqrt{21}\,d\tau = \tau^3\sqrt{21}\Big|_0^t = t^3\sqrt{21} - 0 = \boxed{t^3\sqrt{21}}$

total length $s(1) = 1^3\sqrt{21} = \sqrt{21}$

Because velocity vector $(\langle 1,-2,4 \rangle)$ is always pointing in the same direction, it's a line.

start point → $\vec{OP}$ = when $t=0 \Rightarrow \langle 3,6,1 \rangle$ $\quad$ $\vec{OQ}$ ← stop point: when $t=1 \Rightarrow \langle 4,4,5 \rangle$

$$r(t) = \vec{OP} + t \cdot \vec{PQ} \qquad \vec{PQ} = \langle 4-3, 4-6, 5-1 \rangle$$
$$= \langle 3,6,1 \rangle + t\langle 1,-2,4 \rangle \qquad = \langle 1,-2,4 \rangle$$
$$= \langle 3+t,\ 6-2t,\ 1+4t \rangle$$

velocity $r'(t) = \langle 1,-2,4 \rangle$

$|v(t)|$

$$s(t) = \int_0^t \sqrt{21}\,d\tau \Rightarrow s = \sqrt{21}\,t \Rightarrow t = \frac{s}{\sqrt{21}}$$

$r(t) = \langle 3+t,\ 6-2t,\ 1+4t \rangle$

$r(s) = \langle 3+\left(\frac{s}{\sqrt{21}}\right),\ 6-2\left(\frac{s}{\sqrt{21}}\right),\ 1+4\left(\frac{s}{\sqrt{21}}\right) \rangle \Rightarrow$ unit speed $= |r'(s)| = 1$

EX) Helix $r(t) = \langle \cos 2t, \sin 2t, 3t \rangle \quad 0 \le t \le \pi$

 circle spiraling upward

$v(t) = r'(t) = \langle -2\sin 2t, 2\cos 2t, 3 \rangle$

speed: $|v(t)| = \sqrt{4(\sin 2t)^2 + 4(\cos 2t)^2 + 9}$

$= \sqrt{4+9} = \sqrt{13}$

Distance = (speed)(time) ← speed is constant

$s(t) = (\sqrt{13})(t - 0)$

total length = $\sqrt{13}\,\pi$

$t = \frac{s}{\sqrt{13}}$

$r(s) = \langle \cos\left(\frac{2}{\sqrt{13}}s\right), \sin\left(\frac{2}{\sqrt{13}}s\right), \left(\frac{3}{\sqrt{13}}s\right) \rangle$

EX) Ellipse $r(t) = \langle \cos t, \sin t, 1 - \cos t \rangle \quad 0 \le t \le 2\pi$

(intersection of $x + z = 1$, $x^2 + y^2 = 1$ (plane & cylinder))

$v(t) = r'(t) = \langle \sin t, \cos t, \sin t \rangle$

speed: $|v(t)| = \sqrt{(\sin t)^2 + (\cos t^2) + \sin t^2}$

$= \sqrt{1 + (\sin t^2)}$

Arc length: $\int_0^{2\pi} \sqrt{1 + \sin^2 t}\, dt$

Arc length parameter $s(t) = \int_0^t \sqrt{1 + \sin^2 \tau}\, d\tau$ Yikes!

Just leave it this way b/c it gets hard!

EX) $r(t) = \langle t, t\sin(\frac{\pi}{t}) \rangle$ for $0 < t \leq 1$

$r(0) = \langle 0, 0 \rangle$

$v(t) = r'(t) = \langle 1, \sin(\frac{\pi}{t}) + t\cos(\frac{\pi}{t}) \frac{d}{dt}(\frac{\pi}{t}) \rangle$

$= \langle 1, \sin(\frac{\pi}{t}) - \frac{\pi}{t}\cos(\frac{\pi}{t}) \rangle$

speed $|v(t)| = \sqrt{1 + [\sin(\frac{\pi}{t}) - \frac{\pi}{t}\cos(\frac{\pi}{t})]^2}$

$$\int_0^1 |v(t)|\,dt = \infty$$

* Non-rectifiable curve! No definite length

Going back...

EX 1) The line: $v(t) = \langle 2t^2, -4t^2, 8t^2 \rangle$

$|v(t)| = 2t^2\sqrt{21}$

unit tangent vector:

$$T = \frac{v(t)}{|v(t)|} = \left\langle \frac{1}{\sqrt{21}}, \frac{-2}{\sqrt{21}}, \frac{4}{\sqrt{21}} \right\rangle$$

EX 2) The helix: $v(t) = \langle -2\sin 2t, 2\cos 2t, 3 \rangle$

$|v(t)| = \sqrt{13}$

$$T = \left\langle \frac{-2}{\sqrt{13}}\sin 2t, \frac{2}{\sqrt{13}}\cos 2t, \frac{3}{\sqrt{13}} \right\rangle$$

EX 3) The ellipse: $v(t) = (-\sin t, \cos t, \sin t)$

$|v(t)| = \sqrt{1+\sin^2 t}$

$$T = \left\langle \frac{-\sin t}{\sqrt{1+\sin^2 t}}, \frac{\cos t}{\sqrt{1+\sin^2 t}}, \frac{\sin t}{\sqrt{1+\sin^2 t}} \right\rangle$$