9/26/14

Manifolds & tangent spaces

Describe these with functions & derivatives

$x^2+y^2=1 \rightarrow$ 1 sphere

$x^2+y^2+z^2=1 \rightarrow$ 2 sphere

$w^2+x^2+y^2+z^2=1 \rightarrow$ 3 sphere

Manifold $\longrightarrow$ functions (lots of choices)

functions $\longrightarrow$ manifolds (3 choices)

$f: \mathbb{R}^n \rightarrow \mathbb{R}^m$

graph of f $\quad \{(p, f(p)) \mid p \text{ is in } \mathbb{R}^n\} \subseteq \mathbb{R}^{n+m}$ (locally not only locally)

image of f $\quad \{f(p) \mid p \text{ is in } \mathbb{R}^m\} \subseteq \mathbb{R}^m$ (All manifolds are images locally)

levelset of f $\quad \{p \mid f(p)=c \text{ a constant}\} \subseteq \mathbb{R}^n$

$\hookrightarrow$ (some manifolds are levelsets of a function)

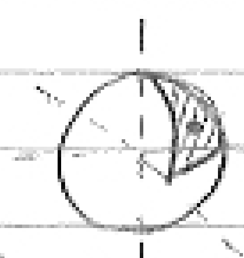
levelset

Sphere of radius 1 $\qquad x^2+y^2+z^2=1$

$f(x,y,z) = x^2+y^2+z^2$

$M = f(x,y,z) - 1$

in 1^{st} octant



image

$g(u,v) = (\cos u \sin v, \sin u \sin v, \cos v) \qquad 0 < u, v < \pi/2$

$(\cos u \sin v)^2 + (\sin u \sin v)^2 + (\cos^2 v)$

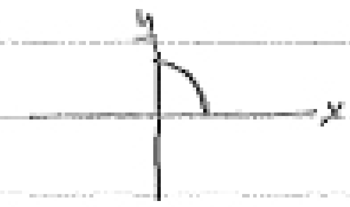
$= \cos^2 u \sin^2 v + \sin^2 u \sin^2 v + \cos^2 v$

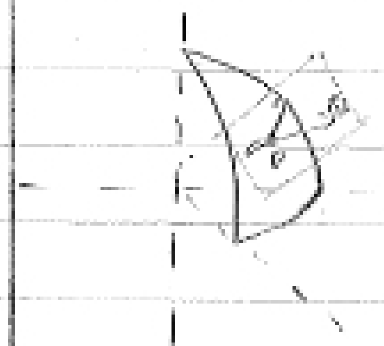
$= (\cos^2 u + \sin^2 u)\sin^2 v + \cos^2 v = 1 \rightarrow$ it's on the sphere!

graph

$h(x,y) = \sqrt{1-x^2-y^2}$

$(x, y, \sqrt{1-x^2-y^2})$





tangent space

$\vec{PQ}$

derivative.

image: tangent vectors are multiples of derivatives.

graph: tangent space is graph of linearization

$$L(Q) = f(p) + f'(P)\cdot(Q-P)$$

$f: \mathbb{R}^n \to \mathbb{R}^1$

levelsets $\to$ $\{P \mid f(P) = c \text{ a constant}\} \subseteq \mathbb{R}^n$

tangent vectors $\vec{PQ}$ are the ones killed by $f'(P)$.

EX) $f: \mathbb{R}^2 \to \mathbb{R}^1$

$f(x,y) = x^2y + y$

$P = (1,2)$

$f(1,2) = 4$

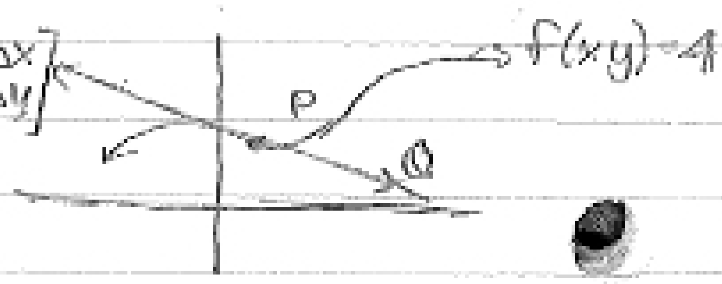
level curve:

$f(x,y) = f(1,2)$

$x^2y + y = 4$

$y = 4/(1+x^2)$

$$f(1+\Delta x, 2+\Delta y) \approx f(1,2) + f'(1,2)\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}$$
$$= 4 + (2xy \quad x^2+1)\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}$$
$$= 4 + \begin{bmatrix}4 & 2\end{bmatrix}\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}$$
$$= 4 + \underbrace{4\Delta x + 2\Delta y}_{0}$$



Want Q so that $f'(P)\cdot\vec{PQ} = 0$

Theorem: If M is defined by $f(x, y, \cdots) = c$ then the tangent space of $(x_0, y_0, \cdots)$ is defined by $f'(x_0, y_0, \cdots) \begin{pmatrix} x - x_0 \\ y - y_0 \\ \vdots \end{pmatrix} = 0$

back to before:

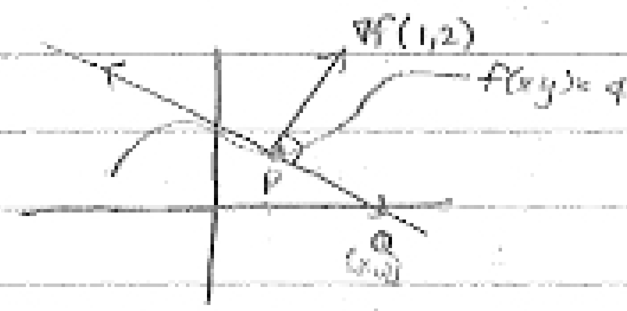
$$4 + \begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \end{bmatrix} = 0$$

equation of tangent line $\Rightarrow 4(x-1) + 2(y-2) = 0$

$\nabla f(x, y) = \langle 2xy, x^2+1 \rangle$

$\nabla f(1,2) = \langle 4, 2 \rangle$

$\langle 4, 2 \rangle \cdot \langle x-1, y-2 \rangle = 0$



$f(x, y, z) = x^2 + y^2 + z^2$

$M = f(x, y, z) = 1 \qquad P = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$

$\nabla f(x, y, z) = \langle 2x, 2y, 2z \rangle$

$\nabla f(1/2, 1/2, 1/\sqrt{2}) = \langle 1, 1, \sqrt{2} \rangle$

$Q - P = \langle x - 1/2, y - 1/2, z - \sqrt{2} \rangle$

tangent plane (Q on plane)

$\nabla f(P) \cdot \overline{PQ} = 0$

$\langle 1, 1, \sqrt{2} \rangle \cdot \langle x - 1/2, y - 1/2, z - \sqrt{2} \rangle = 0$

$(x - 1/2) + (y - 1/2) + \sqrt{2}(z - 1/\sqrt{2}) = 0$