Notes on Nash Equilibrium †

ECON 201B - Game Theory

Guillermo Ordoñez
UCLA

January 19, 2007

1 Normal (or Strategic) Form Games

A game in strategic (normal) form is:

- A finite set of players $i = 1, ..., N$
- For each player i, a non-empty set of actions S_i (pure strategies)
- For each player i, a utility or payoff (as a function of strategy profiles) $u_i(s) : S \equiv \times_{i=1}^{N} S_i$ (usually interpreted as von Neumann-Morgenstern utility)

A useful notation is
$s_{-i} \in S_{-i} \equiv \times_{j \neq i} S_j$, the "player i's opponents" strategies profile
$s = (s_i, s_{-i}) \in S$
$u_i(s) = u_i(s_i, s_{-i}) = u_i(s_i | s_{-i})$

As can be seen u_i depends on actions of ALL players and can be extended to mixed strategies and mixed strategies profiles, defined as $\sigma_i \in \Sigma_i \equiv P(S_i)$, such that
$\sigma_{-i} \in \Sigma_{-i} \equiv \times_{j \neq i} \Sigma_j$
$\sigma = (\sigma_i, \sigma_{-i}) \in \Sigma$
$u_i(\sigma) = u_i(\sigma_i, \sigma_{-i}) = u_i(\sigma_i | \sigma_{-i})$

the expected utility or payoff can be expressed as $u_i(\sigma) = \sum_{s \in S} \left[u_i(s) \prod_{j=1}^{N} \sigma_j(s_j) \right]$
(if S is finite, this is a polynomial in σ (hence continuous), affine in σ_i)

[0] † These notes were prepared as a back up material for TA session. If you have any questions or comments, or notice any errors or typos, please drop me a line at guilord@ucla.edu

2 Dominated Strategies

Definition 1 *A pure strategy s_i is **strictly dominated** for player i if there exists $\sigma_i' \in \Sigma_i$ such that*

$$u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}$$

Definition 2 *A pure strategy s_i is **weakly dominated** for player i if there exists $\sigma_i' \in \Sigma_i$ such that*

$$u_i(\sigma_i', s_{-i}) \geq u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}$$

and strict for at least one $s_{-i} \in S_{-i}$

In words, a strictly dominated strategy is just an strategy that would not be used NO MATTER how opponents play.

When applying a iterated elimination of dominated strategies it's necessary to check domination of all strategies after each round of elimination. Typically it is the case that a strategy not dominated in the original game is dominated after the elimination of some of the opponents' strategies.

3 Nash Equilibrium

3.1 Definition

A Nash Equilibrium (NE) is a profile of strategies such that **each** player's strategy is an optimal response to the other players' strategies.

Definition 3 *A mixed-strategy profile σ is a Nash Equilibrium if, for **each** i and for all $\sigma_i' \neq \sigma_i$*

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

A pure-strategy Nash Equilibrium is a pure-strategy profile that satisfies the same conditions.

A Nash Equilibrium is strict if each player has a unique best response to his rivals' strategies. That is, s is a strict equilibrium if, for each i and for all $s_i' \neq s_i$, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$. As denoted by Fudenberg and Tirole, by definition, a strict equilibrium is necessarily a pure-strategy equilibrium.

3.2 Computation (The recipe)

How to compute Nash Equilibria?

a) Construct the Normal form of the game (where players, actions and payoffs are the components as defined at the beginning).

b) Find the best responses that each player have against the opponents' pure strategies

For each player take each possible combination of opponents' pure-strategies and choose the own strategy that maximizes payoffs in each case.

c) Proceed iteratively to eliminate all strictly dominated pure strategies

Eliminate strategies that are never best response, no matter what the others are doing, (i.e. strategies that will never be used). Since this follows a iterative elimination it's necessary to check all the strategies after each round of elimination.

d) Check if there is a Nash equilibrium in pure strategies

A NE in pure strategies is the combination of strategies where all the best responses in pure strategies concide. In this case there are no incentives for anybody to deviate.

e) Find all mixed strategy equilibria by listing all possible combinations for the support of mixed strategies for each player and checking the necessary and sufficient conditions for a Nash equilibrium in mixed strategies.

3.3 Examples

3.3.1 Example 1: Matching Pennies (ROW chooses row, COL chooses column)[1]

	L	R
U	$1,-1$	$-1,1$
D	$-1,1$	$1,-1$

Following the steps of the recipe above,

a) Given by the chart below

b) Best responses in bold for each possible play of the opponent.

	L	R
U	$\mathbf{1},-1$	$-1,\mathbf{1}$
D	$-1,\mathbf{1}$	$\mathbf{1},-1$

[1] This is a zero sum game, (zero sum games are characterized by the fact that $\sum_{i=1}^{N} u_i(s) = 0$, for all $s \in S$). In fact, as denoted by FT, the key is that the sum of the utilities is a constant and setting the constant equal to 0 is just a normalization.