15-251
Great Theoretical Ideas in Computer Science

Cantor's Legacy: Infinity and Diagonalization

Lecture 23 (April 7, 2009)

$$\infty$$

Ideas from the course

Induction
Numbers, Number theory and Algebra
Representation
Finite Counting and Probability
Automata and Computation

A hint of the infinite
- Infinite row of dominoes
- Infinite sums (formal power series)
- Infinite choice trees, and infinite probability

Infinite RAM Model

Platonic Version:
One memory location for each natural number 0, 1, 2, …

Aristotelian Version:
Whenever you run out of memory, the computer contacts the factory. A maintenance person is flown by helicopter and attaches 1000 Gig of RAM and all programs resume their computations, as if they had never been interrupted.

The Ideal Computer: no bound on amount of memory no bound on amount of time

Ideal Computer is defined as a computer with infinite RAM.

You can run a Java program and never have any overflow, or out of memory errors.

An Ideal Computer

It can be programmed to print out:

2: 2.0000000000000000000000…
1/3: 0.33333333333333333333…
ϕ: 1.6180339887498948482045…
e: 2.7182818284559045235336…
π: 3.14159265358979323846264…

Printing Out An Infinite Sequence..

A program P prints out the infinite sequence
$s_0, s_1, s_2, \ldots, s_k, \ldots$
if when P is executed on an ideal computer, it outputs a sequence of symbols such that

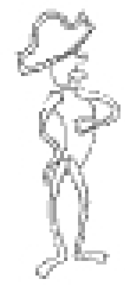
-The k^{th} symbol that it outputs is s_k

-For every $k \in \mathbb{N}$, P eventually outputs the k^{th} symbol. I.e., the delay between symbol k and symbol k+1 is not infinite.

Computable Real Numbers

A real number R is computable if there is a program that prints out the decimal representation of R from left to right.

Thus, each digit of R will eventually be output.



Are all real numbers computable?

Describable Numbers

A real number R is describable if it can be denoted unambiguously by a finite piece of English text.

2: "Two."
π: "The area of a circle of radius one."

Are all real numbers describable?

Is every computable real number, also a describable real number?

And what about the other way?

Computable R: some program outputs R
Describable R: some sentence denotes R

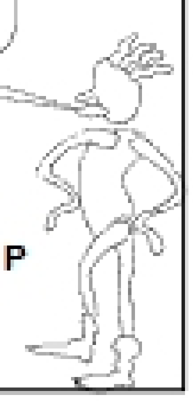
Computable ⇒ describable

Theorem:
Every computable real is also describable

Proof:
Let R be a computable real that is output by a program P. The following is an unambiguous description of R:

"The real number output by the following program:" P

MORAL: A computer program can be viewed as a description of its output.



Syntax: The text of the program
Semantics: The real number output by P

Are all reals describable?
Are all reals computable?

We saw that computable ⇒ describable but do we also have describable ⇒ computable?

Questions we will answer in this (and next) lecture…

Correspondence Principle

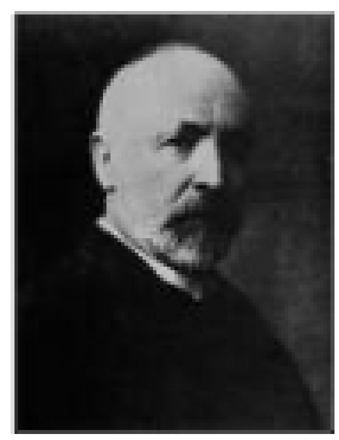
If two finite sets can be placed into 1-1 onto (bijective) correspondence, then they have the same size.

Correspondence Definition

In fact, we can use the correspondence as the definition:

Two finite sets are defined to have the same size if and only if they can be placed into 1-1 onto (bijective) correspondence.

Georg Cantor (1845-1918)



Cantor's Definition (1874)

Two sets are defined to have the same size (→ cardinality) if and only if they can be placed into 1-1 onto correspondence.

If there exists a bijection between them.

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