Math 216 Differential Equations

Lab 5: Nonlinear Systems

Goals

In this lab you will use the `pplane8` program to study two nonlinear systems by direct numerical simulation.

The first model, from population biology, displays interesting nonlinear oscillations (so-called *limit cycles*). They are isolated closed orbits like the red circular one in Figure 1.

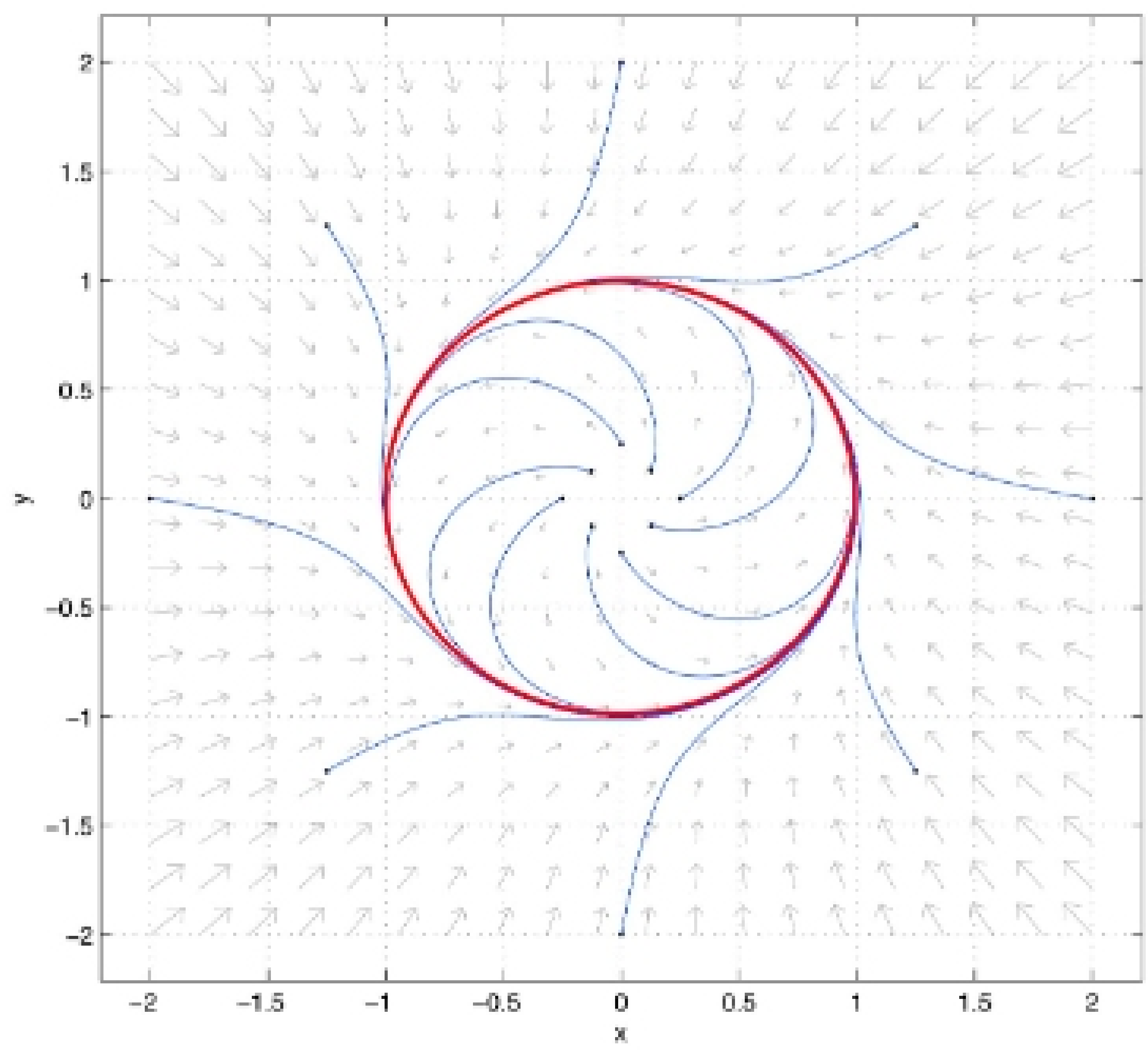

Figure 1: Limit cycle

If one starts on the red circle, one keeps going around on it counterclockwise. The limit cycle in the picture is an *attractor*, that is, if the initial point is sufficiently close to the circle then its orbit will approach the circle.

The second model is a system whose solutions depend on a parameter, which thus is an illustration of a system with a *bifurcation*.

Neither of these systems is described by exactly solvable systems of differential equations. Although much may be learned from strictly theoretical analyses, we must ultimately rely on computational methods to extract quantitative predictions from these systems.

Model 1: Predator-prey interactions

In class we considered a model of predator-prey species interactions known as the Lotka-Volterra model (referred to in Section 6.3 as the *predator-prey system*). If x describes the size of a population of rabbits and y describes a population of foxes (which feed on the population of rabbits) then the Lotka-Volterra model of their interactions says that there are positive constants a, b, c, d so that

$$\frac{dx}{dt} = x(a - by)\,, \tag{1}$$
$$\frac{dy}{dt} = y(-c + dx)\,. \tag{2}$$

While relatively simple, this model has some shortcomings. For example, it suggests that predation is proportional to the number of prey—so that if $y = 10$ foxes and $x = 100,000$ rabbits, the foxes will eat 10 times more than if there were $y = 10$ foxes and $x = 10,000$ rabbits. In reality, we would expect that a small number of foxes surrounded by that a large number of rabbits would eat the same amount independent of the number of rabbits.

A more reasonable model is

$$\frac{dx}{dt} = x(1 - x) - \frac{k\,xy}{k\,x + 1}\,, \tag{3}$$
$$\frac{dy}{dt} = ry\left(1 - \frac{y}{x}\right)\,, \tag{4}$$

where r is a positive constant. In this model, in the absence of predators the prey satisfies the logistic equation with equilibrium population $x = 1$. In the presence of predators, prey is consumed at a rate $kxy/(kx + 1)$. That is, if x is large compared to $1/k$, then $kxy/(kx + 1) \approx y$, and the predators consume prey at a rate proportional to the predator population. On the other hand, if x is small compared to $1/k$ then $kxy/(kx + 1) \approx kxy$ and only then do the predators consume prey at a rate proportional to xy as in the Lotka-Volterra model. (Note: We have already scaled the variables and chosen some parameter values in equations (3) and (4). The general version of the model would have many more parameters.)

Model 2: Bifurcations

We saw in §2.2 that the equilibrium solutions of an autonomous first-order differential equation that includes a parameter may change if the parameter changes. This is called a *bifurcation*. Similarly, for nonlinear autonomous systems we may see that the equilibrium solutions change when a parameter in the system changes: what the system does may depend on the mass of some component of the system, the stiffness of a spring, the length of a lever, the resistance of an electronic component, and so on. In this section of the lab you will observe in a very simple case how the number of equilibrium points of a system of differential equations changes as a parameter varies. Note that we call the parameter value at which the number or stability of equilibrium points changes the *bifurcation point*.

Bifurcations are significant because a change in the number of equilibrium points, or in the stability of the equilibria, usually leads to a dramatic qualitative change in the behavior of the non-equilibrium solutions as well. In the following we shall study the nonlinear system of differential equations

$$\frac{dx}{dt} = x - xy\,, \tag{5}$$

$$\frac{dy}{dt} = x + ay\,. \tag{6}$$

This system exhibits a bifurcation as the parameter a varies.

Prelab assignment

Before arriving in the lab, answer the following questions. You will need your answers in lab to work the problems, and your recitation instructor may check that you have brought them. These problems are to be handed in as part of your lab report.

1. In the modified predator-prey system (equations (3) and (4), let $k = 7$. Find all of the critical points (fixed points, or equilibrium points). Calculate the numerical value of the "coexistence point," i.e. the critical point for which both x and y are positive.

2. There are two critical values of r at which the behavior of the system changes dramatically. These are $r_{c1} = 0.15843239\ldots$ and $r_{c2} = 2.41985814\ldots$. For $r < r_{c1}$, $r_{c1} < r < r_{c2}$ and $r > r_{c2}$ there are three behaviors: in two of these regions the coexistence point is stable, and in two it is a spiral (either stable or unstable). Which happens for which values of r? (Hint: plug in two different values of r and do a critical point analysis for each.)

3. Find the equilibrium points for the second system; that is, for equations (5) and (6). Note that your solutions will depend on the parameter a. You should find two equilibrium points.

4. Find the Jacobian for the second system. Find the eigenvalues for the two equilibrium points in terms of the parameter a. At what values of a do the types of the equilibrium points change? (You should find two values of a where this is the case.)

In the lab

We will be using Matlab with the program `pplane8` to study phase portraits for nonlinear systems. The program `pplane8` is an analogue of `dfield8` that we used in Lab 1 that is specially adapted to systems of two differential equations. Using `pplane8`, we can let Matlab handle the numerical approximation of solutions and instead focus on the meaning of the solutions and the way that solutions change as parameters are varied.