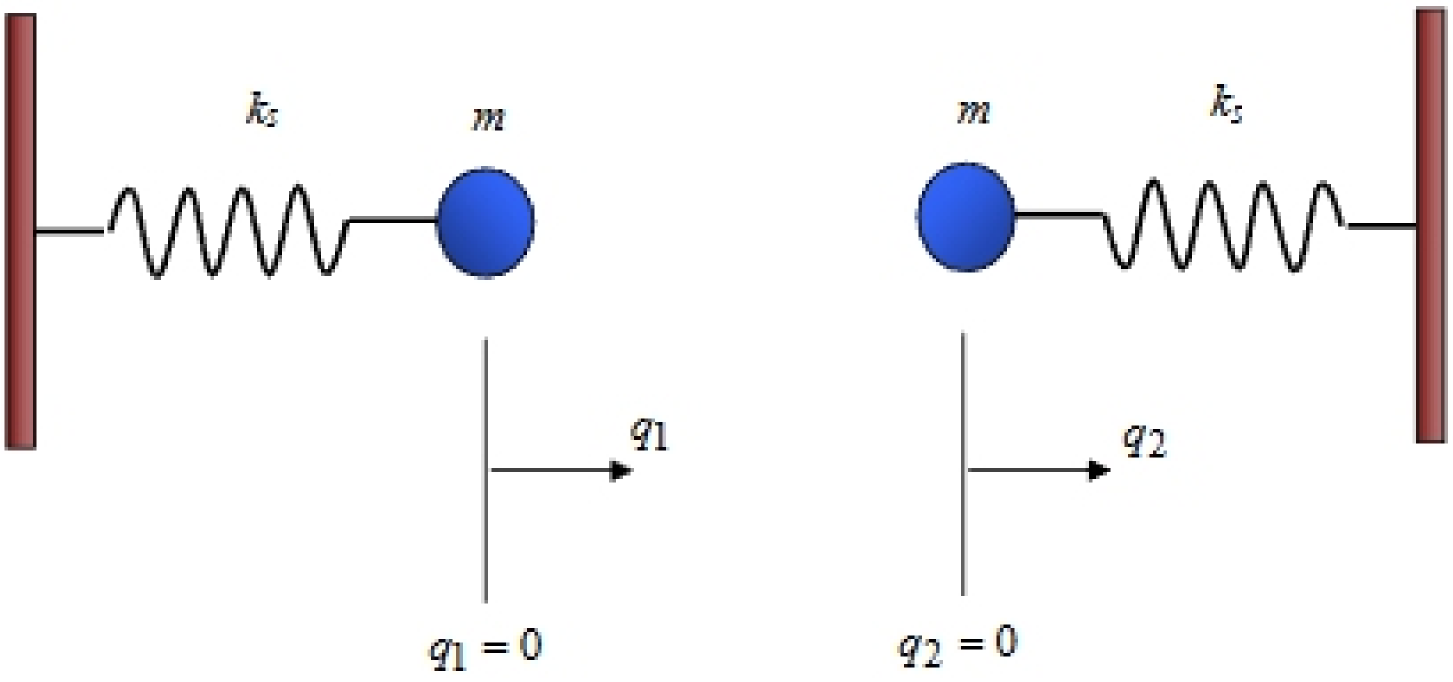
Two Coupled Oscillators / Normal Modes

Overview and Motivation: Today we take a small, but significant, step towards wave motion. We will not yet observe waves, but this step is important in its own right. The step is the coupling together of two oscillators via a spring that is attached to both oscillating objects.

Key Mathematics: We gain some experience with coupled, linear ordinary differential equations. In particular we find special solutions to these equations, known as normal modes, by solving an eigenvalue problem.

I. Two Coupled Oscillators

Let's consider the diagram shown below, which is nothing more than 2 copies of an harmonic oscillator, the system that we discussed last time. We assume that both oscillators have the same mass m and spring constant k_s. Notice, however, that because there are two oscillators each has it own displacement, either q_1 or q_2.



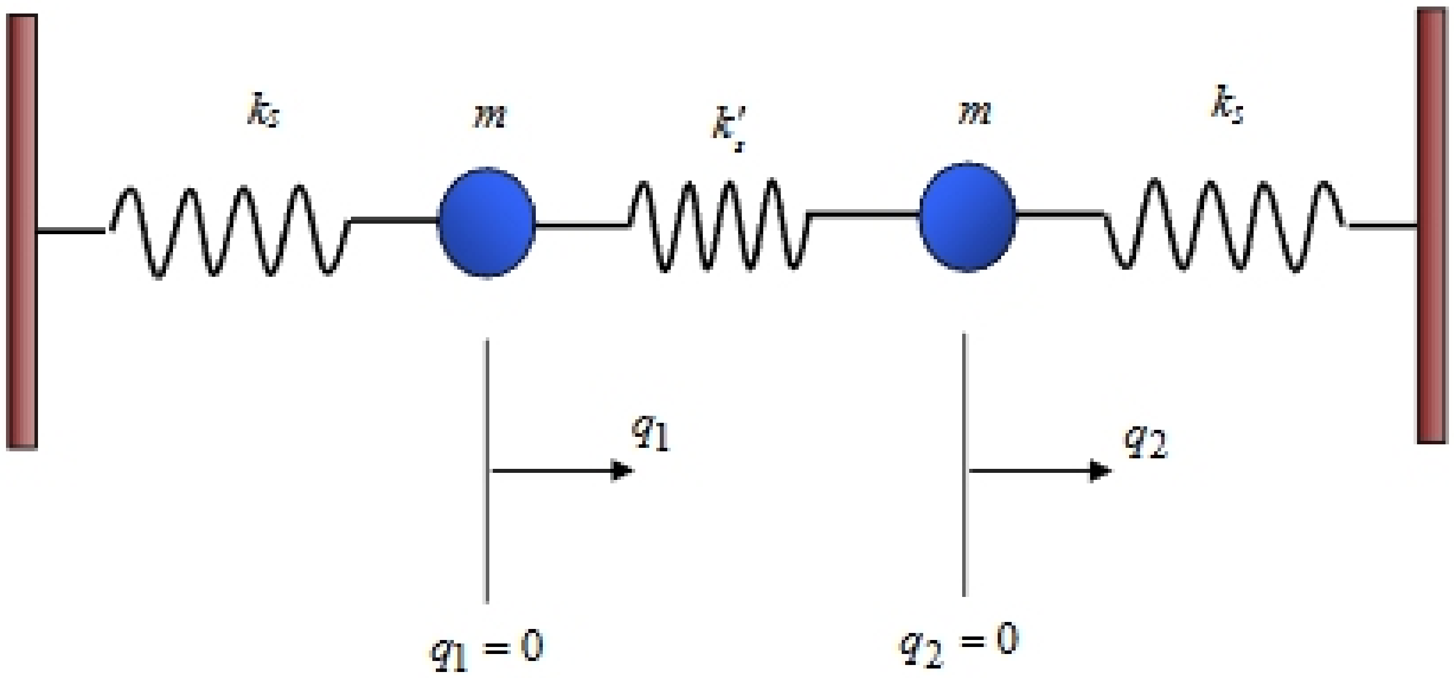
Based on the discussion last time you should be able to immediately write down the equations of motion (one for each oscillating object) as

$$\ddot{q}_1 + \tilde{\omega}^2 q_1 = 0 \text{, and} \tag{1a}$$

$$\ddot{q}_2 + \tilde{\omega}^2 q_2 = 0 , \tag{1b}$$

where $\tilde{\omega}^2 = k_s / m$. As we saw last time, the solution to each of theses equations is harmonic motion at the (angular) frequency $\tilde{\omega}$. As should be obvious from the picture, the motion of each oscillator is independent of the other oscillator. This is also reflected in the equation of motion for each oscillator, which has nothing to do with the other oscillator.

Let's now make things a bit more interesting by adding in another spring that connects the two oscillating objects together, as illustrated in the following picture. To make things even more interesting we assume that this new spring has a different constant k_s'. However, to keep things simple we assume that the middle spring provides no force if $q_1 - q_2 = 0$. That is, this spring is neither stretched or compressed if its length is equal to the its length when two objects are at equilibrium.



Thinking about this picture we should realize that the two equations of motion will no longer be independent. That is, the equation of motion for the first object will depend (somehow) upon what the second object is doing, and vice versa.

Let's use Newton's second law to write down the equation motion for each object. Recall that Newton's second law for either object $(i = 1,2)$ can be written as

$$\ddot{q}_i = \frac{F_i}{m}, \tag{2}$$

where F_i is the net force on object i. The tricky part, if there is a tricky part, is to determine the sum F_i on each object. The net force on the first object comes from the spring on the left and the spring in the middle. With a little thought you should realize that this net force F_1 is

$$F_1 = -k_s q_1 - k'_s(q_1 - q_2). \tag{3a}$$

Make sure that you understand the signs of all the term on the rhs of this equation. Notice that the force provided by the middle spring depends not only on the first object's displacement but also on the second object's displacement. Similarly, the net force on the second object is

$$F_2 = -k_s q_2 - k'_s(q_2 - q_1). \tag{3b}$$

Substituting these two forces into Eq. (2), once for each object, we obtain the two equations of motion,

$$\ddot{q}_1 + \tilde{\omega}^2 q_1 + \tilde{\omega}'^2 (q_1 - q_2) = 0 \tag{4a}$$

for the first object and

$$\ddot{q}_2 + \tilde{\omega}^2 q_2 + \tilde{\omega}'^2 (q_2 - q_1) = 0 \tag{4b}$$

for the second. Here $\tilde{\omega}'^2 = k'/m$. Given the symmetry of the problem, it might not surprise you that you can obtain one equation of motion from the other with the transformation $1 \leftrightarrow 2$ in the subscripts that label the objects.

So now we have a considerably more complicated problem: as expected from looking at the drawing above, the equation of motion for each object depends upon what the other object is doing. Specifically, each equation of motion depends upon the displacement of the other object.

II. Normal Modes

A. Harmonic Ansatz

So what are the solutions to these differential equations? Well, we will eventually write down the general solution (next lecture). But right now we are going to look at a special class of solutions known as normal-mode solutions, or simply, normal modes. A **normal mode** is a solution in which both masses harmonically oscillate at the same frequency. We state why these special solutions are extremely useful at the end of the lecture. For now let's see if we can find them. We use the complex form of harmonic motion and write

$$q_1(t) = q_{01} e^{i\Omega t} \text{ and} \tag{5a}$$