Proof by Contradiction Notes

RUBRIC

- "We proceed by contradiction."
- State an assumption
- Assumption is theorem negated
- Instatiate a counterexample, given negated theorem
- Any algebra is correct
- Any appeal to definition/theorem/etc. is correct and appropriately applied
- Correct derivation of a contradiction
- Concludes that contradiction means original theorem is true
- Written in English, not just symbols

FORMAT

We proceed by contradiction.
Assume <u>(theorem negated assumption)</u>.
Then, <u>(instantiate a counterexample)</u>.
By ..., that means .... (correct appeal to definitions/theorems/etc.).
But, in the case provided, ... (correct derivation of a contradiction using theorems).
Therefore, since this assumption led to a contradiction, ... (conclude the contradiction/reaffirm original theorem).

- Use specific case when theorem involves "for all" ($\forall$)

Number Theory Notes

Factors

- a is a factor of b iff b can be evenly divided by a
    - That is, for some non-zero integer $ak = b$

Primes

- A prime is a number greater than 1 that is divisible only by 1 and itself

Unique Factorization / Fundamental Theorem of Arithmetic

- For all natural numbers greater than 1, there exists a factorization of n such that all factors are prime and the factorization is unique
- Multiplicity of factor = number of occurrences

Greatest Common Divisor

- A common divisor of a and b is a number that divides them both
- The greatest common divisor of a and b is the biggest number that divides them both

Relatively Prime / Coprime

- Two positive integers greater than or equal to 1 are relatively prime iff $\gcd(x, y) = 1$

Rational Numbers

- Represented by $\mathbb{Q}$, which stands for quotient
    - All rational numbers can be represented by a quotient of two integers
    - $x \in \mathbb{Q}$ iff $x = \frac{a}{b}$, where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$

1. We proceed by contradiction.
2. Assume the opposite of the statement.
3. By the definition ..., ...
4. If we take the counterexample, then... (algebra)
   - Elaborate if necessary
5. However, ..., provide contradiction.

Example — Format points

1. We proceed by contradiction.
2. Assume that $(|S \cup \{x\}| = |S|) \land \neg(x \in S)$.
3. By the definition of union, we know that for any sets A and B, $|A \cup B| = |A| + |B| - |A \cap B|$. This means that the cardinality of $S \cup \{x\}$ is $|S| + 1 - 0 = |S| + 1$
4. Therein lies the contradiction. Our assumption states that $|S \cup \{x\}|$ is $|S|$, but we have just derived that the cardinality of $S \cup \{x\}$ is $S + 1$.
5. This means we can conclude that $(|S \cup \{x\}| = |S|) \to (x \in S)$