SOLUTION KEY / ASSIGNMENT 1 / MATH 420

§1.2 #13 V with the defined operations fails (VS 8):

$(s+t)(a_1,a_2) = ((s+t)a_1, a_2)$

$s(a_1,a_2) + t(a_1,a_2) = (sa_1,a_2)+(ta_1,a_2) = ((s+t)a_1, a_2^2)$

Eg $s=1$, $t=0$, $(a_1,a_2)=(1,2)$ we have

$(1+0)(1,2) = (1,2)$

$1(1,2)+0(1,2) = (1,2)+(0,2) = (1,4)$

#16 $V = M_{m\times n}(\mathbb{R})$, $F = \mathbb{Q}$ (rationals).

V is a v.s. over $\mathbb{Q}$ since $\mathbb{Q} \subseteq \mathbb{R}$ and since $\mathbb{Q}$ is a field

#18 Addition is not commutative, so NO:

$(1,2)+(3,4) = (1+6, 2+12) = (7,14)$

$(3,4)+(1,2) = (3+2, 4+6) = (5,10)$

§1.3 #5 $(A+A^t)_{ij} = A_{ij} + (A^t)_{ij} = A_{ij} + A_{ji}$

and $([A+A^t]^t)_{ij} = (A+A^t)_{ji} = A_{ji} + A_{ij}$

so $(A+A^t)^t = A+A^t$

#8 (b) $W_2 = \{(a_1,a_2,a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2\}$
<u>Not</u> a subspace of $\mathbb{R}^3$ since $\theta \notin W_2$

(d) $W_4 = \{(a_1,a_2,a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - a_3 = 0\}$
is a subspace by Thm 1.3

(f) W_6 is <u>not</u> a subspace because it is <u>not</u> closed under addition:

$(1, 2, \sqrt{7/6}), (0, 1, 1/\sqrt{2}) \in W_6$ but their sum $\notin W_6$.

#10 $\theta \notin W_2$ so W_2 is <u>not</u> a subspace. W_1 satisfies the conditions of Thm 1.3 so it is a subspace.

#12 Apply Thm 1.3 and matrix operations

#19 Let $W_1 \leq V$, $W_2 \leq V$ and let $W = W_1 \cup W_2$

($\Rightarrow$) Suppose W is a subspace and, by way of contradiction, $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$. Then $\exists x \in W_1 \cap W_2^c$, $\exists y \in W_2 \cap W_1^c$.
Consider now $x+y$ which belongs to W since $x, y \in W$ and since W is a subspace. There are two cases now:
(1) $x+y = z \in W_1$. Then $y = z - x \in W_1$, contradiction.
(2) $x+y = z \in W_2$. Then $x = z - y \in W_2$, contradiction.
So, either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

($\Leftarrow$) Suppose $W_1 \subseteq W_2$ (in which case $W_1 \cup W_2 = W_2$) or $W_2 \subseteq W_1$ (in which case $W_1 \cup W_2 = W_1$). In either case $W_1 \cup W_2 \leq V$.

\# 31 V is a v.s., $W \leq V$.

(a) ($\Rightarrow$) Suppose $v+W \leq V$. Thus there exists $w \in W$ such that $v+w=\theta$. Since V is a v.s. it follows that $w=-v \in W$; since W is a subspace it follows that $v \in W$.

($\Leftarrow$) Suppose $v \in W$. Then $v+W \subseteq W$. Also, $\forall x \in W$, $x = v+(x-v) \in v+W$, i.e. $W \subseteq v+W$. So $v+W = W \leq V$.

(b) $v_1+W = v_2+W \iff \exists\, w_1, w_2 \in W$ such that $v_1+w_1 = v_2+w_2$

$\iff v_1 - v_2 = w_2 - w_1 \in W$