12, 20, 21

$$A = \begin{pmatrix} 1 & 2 & -4 & 3 & 3 \\ 5 & 10 & -9 & -7 & 8 \\ 4 & 8 & -9 & -2 & 7 \\ -2 & -4 & 5 & 0 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -4 & 3 & 3 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 + 4R_2} \begin{pmatrix} 1 & 2 & 0 & -5 & 3 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-\frac{1}{5} R_3} \begin{pmatrix} 1 & 2 & 0 & -5 & 3 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 - 3R_2} \begin{pmatrix} 1 & 2 & 0 & -5 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$$ columns 1, 3 and 5 are basic, i.e. lin. independent $\Rightarrow$ basis for the column space is

$$\underline{B_{col}} = \left\{ \begin{pmatrix} 1 \\ 5 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} -4 \\ -9 \\ -9 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 8 \\ 7 \\ -6 \end{pmatrix} \right\}.$$

Clearly, $\dim B_{col} = 3$.

Now, solution space of the homogeneous matrix equation $Ax = 0$ gives $N(A)$:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2x_2 + 5x_4 \\ x_2 \\ 2x_4 \\ x_4 \\ 0 \end{pmatrix}$$

$$= x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 5 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$

So we see that $N(A)$ is spanned by the set $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right\}$, which forms ~~the~~ a basis for $N(A)$.

$\dim N(A) = 2$; so that $5 = \dim N(A) + \text{rank}(A) = \#$ of columns of A.

20. By Rank - Nullity theorem, rank $A = 5 - \dim N(A)$. (3)

Since $\dim N(A) = 3$, rank $A = 2$.

21. The dimension of the solution space

$Ax = 0$ is the dimension of the null-space

of A. Since rank $A = 4$, by Rank - Nullity

theorem $\dim N(A) = 6 - 4 = 2$.

§3.1 2, 4, 20, 22

2. $$\begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix} = 0 \cdot \begin{vmatrix} -3 & 0 \\ 4 & 1 \end{vmatrix} - 5 \begin{vmatrix} 4 & 0 \\ 2 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 4 & -3 \\ 2 & 4 \end{vmatrix} =$$

$$= -5 \cdot (4 - 2 \cdot 0) + 1 \cdot (16 + 5) = -20 + 21 = 1.$$

4. $$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} - 3 \cdot \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 5 \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} =$$

$$= (2 - 4) - 3 \cdot (4 - 3) + 5 (8 - 3) =$$

$$= -2 - 3 + 25 = 20.$$