the 3d vector space of (at most) quadratic expressions in a variable

The space of (at most) quadratic functions in a single variable x is a 3-dimensional vector space: the sum of two quadratics is a quadratic whose ordered coefficients are the sum of the individual quadratics, while the ordered coefficients of a scalar multiple of a quadratic are just the scalar times the ordered coefficients of the quadratic. It is natural to order the powers of the variable x in increasing order from 0 to 2 (so that we can easily extend this discussion to any number of powers by adding additional terms, as in Taylor polynomials). Technically a quadratic function must have the coefficient of its squared term be nonzero, otherwise it is a linear function or even a constant function, so we are talking about the space of "at most" quadratic polynomials, namely polynomials of degree at most 2. This will be understood here when we refer to these functions as quadratics.

```
> restart :
  Q1, Q2 := 1 + 2 x + 3 x^2, 2 − x + x^2;
  'Q1 + Q2'= Q1 + Q2,'2 Q1'= 2 Q1
```

$$Q1, Q2 := 1 + 2x + 3x^2, 2 - x + x^2$$

$$Q1 + Q2 = 3 + x + 4x^2, 2\,Q1 = 2 + 4x + 6x^2 \quad \textbf{(1)}$$

To each quadratic corresponds a vector of ordered coefficients, and adding quadratics or scalar multiplying them corresponds directly to the same vector operations on the corresponding vectors:

```
> q1, q2 := ⟨1, 2, 3⟩ , ⟨2, -1, 1⟩ ;
  'q1 + q2'= q1 + q2,'2  q1'= 2 q1
```

$$q1, q2 := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$q1 + q2 = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, 2\,q1 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \quad \textbf{(2)}$$

The ordered set of functions $\{1, x, x^2\}$ is a basis of this vector space since every quadratic can be expressed uniquely as a linear combination of these three functions, with respective coefficients traditonally called $\{c, b, a\}$. These coefficients correspond respectively to the Cartesian coordinates $\{x_1, x_2, x_3\} = \{c, b, a\}$ on R^3, while the basis $\{1, x, x^2\}$ has coefficient vectors $\{ \langle 1, 0, 0\rangle , \langle 0, 1, 0\rangle, \langle 0, 0, 1\rangle \}$ and so corresponds to the usual basis $\{ \vec{i}, \vec{j}, \vec{k} \}$ of $\mathbb{R}^3$. Thus we can picture each quadratic function as a vector (arrow) in space.

The three functions $\{1, x, x^2\}$ are linearly independent functions since if any linear combination of them equals the zero function 0, then all the coefficients have to be zero, and they certainly span the space of all quadratics by definition. Thus they satisfy the two properties of a basis.

However, we could also express any quadratic function in terms of its Taylor polynomial about a value of x different from 0, say $x = 1$. This gives us another set of coefficients which are related to the old coefficients by a linear transformation, equivalently a new basis of this vector space with new coordinates $\{y_1, y_2, y_3\}$:

```
> c + b x + a x^2 = C + B (x − 1) + A (x − 1)^2 :
```

The relationship between them is obtained just by expanding the right hand side (by multiplying it out) and comparing coefficients on both sides of the equation

```
> a x^2 + b x + c = collect(expand(C + B (x − 1) + A (x − 1)^2), x)
```

$$c + bx + ax^2 = Ax^2 + (-2A + B)x + C + A - B \quad \textbf{(3)}$$

so identifying the coefficients of the powers of x on each side gives

> $\langle a, b, c\rangle = \langle A, B-2A, A-B+C\rangle$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} A \\ -2A+B \\ C+A-B \end{bmatrix} \tag{4}$$

The inverse transformation is obtained by simply evaluating the Taylor polynomial centered at $x = 1$ using calculus and comparing coefficients of powers of $(x-1)$ on both sides of the equation:

> $f := x \to a x^2 + b x + c;$
> $f(1), f'(1), f''(1);$
> $f(x) = f(1) + f'(1)\,(x-1) + \frac{1}{2} f''(1)\,(x-1)^2$

$$f := x \to a x^2 + b x + c$$

$$a+b+c,\ 2a+b,\ 2a$$

$$c + b x + a x^2 = a + b + c + (2a+b)\,(x-1) + a\,(x-1)^2 \tag{5}$$

> $C + B\,(x-1) + A\,(x-1)^2 = taylor(a x^2 + b x + c, x = 1, 3)$

$$C + B\,(x-1) + A\,(x-1)^2 = a + b + c + (2a+b)\,(x-1) + a\,(x-1)^2 \tag{6}$$

Matching the coefficients on both sides of the equation:

> $\langle A, B, C\rangle = \langle a, b+2a, c+b+a\rangle$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} a \\ 2a+b \\ a+b+c \end{bmatrix} \tag{7}$$

Rewriting this in ascending order of powers using the more familiar Cartesian coordinate symbols $\langle x_1, x_2, x_3\rangle = \langle c, b, a\rangle$ for the old coefficients and $\langle y_1, y_2, y_3\rangle = \langle C, B, A\rangle$ for the new coefficients, we get the following linear coordinate transformation from the old coordinates to the new coordinates

> $\langle y_1, y_2, y_3\rangle = \langle x_1 + x_2 + x_3, x_2 + 2x_3, x_3\rangle$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1+x_2+x_3 \\ x_2+2x_3 \\ x_3 \end{bmatrix} \tag{8}$$

and from the new coordinates to the old coordinates above

> $\langle x_1, x_2, x_3\rangle = \langle y_1 - y_2 + y_3, y_2 - 2y_3, y_3\rangle$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1-y_2+y_3 \\ y_2-2y_3 \\ y_3 \end{bmatrix} \tag{9}$$

Alternatively we can describe this in terms of he new basis functions $\{1, x-1, (x-1)^2\} = \{1, x-1, 1-2x+x^2\}$ which have old coordinate vectors respectively (which are the columns of the basis changing matrix; careful now $\boldsymbol{B}$ will be a matrix):

> $\langle 1, 0, 0\rangle, \langle -1, 1, 0\rangle, \langle 1, -2, 1\rangle$:
> $\boldsymbol{B} := \langle\langle 1, 0, 0\rangle | \langle -1, 1, 0\rangle | \langle 1, -2, 1\rangle\rangle;$
> $\boldsymbol{B}^{-1}$

$$B := \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad (10)$$

so the expressing the old coordinates in terms of the new coordinates:

> $\langle x_1, x_2, x_3 \rangle = y_1 \langle 1, 0, 0 \rangle + y_2 \langle -1, 1, 0 \rangle + y_3 \langle 1, -2, 1 \rangle$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 - y_2 + y_3 \\ y_2 - 2\,y_3 \\ y_3 \end{bmatrix} \quad (11)$$

Together we have the coordinate transformation and its inverse

> $\langle x_1, x_2, x_3 \rangle = \boldsymbol{B} \, \langle y_1, y_2, y_3 \rangle, \langle y_1, y_2, y_3 \rangle = \boldsymbol{B}^{-1} \langle x_1, x_2, x_3 \rangle$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 - y_2 + y_3 \\ y_2 - 2\,y_3 \\ y_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_2 + 2\,x_3 \\ x_3 \end{bmatrix} \quad (12)$$

As far as linear operations of function addition and scalar multiplication are concerned, instead of working with the abstract vector space of quadratic functions, we can work entirely in ordinary space using coordinates with respect to the "natural basis" of either space which correspond to each other in this obvious correspondence between the two spaces, or some other basis which might be of interest, like the second one we came up with using the Taylor polynomial. This helps us visualize the abstract 3d vector space in terms of the concrete familiar vector space R^3. Visualizable mathematics is always more powerful that symbol pushing.

Here are the old (black) and new (red, blue, green) basis vectors:

```
> with(plots) :
> e1 := arrow(⟨0, 0.03, 0⟩, ⟨1, 0, 0⟩, shape = arrow, color = black, thickness = 2) :
  e2 := arrow(⟨0, 1, 0⟩, shape = arrow, color = black, thickness = 2) :
  e3 := arrow(⟨0, 0, 1⟩, shape = arrow, color = black, thickness = 2) :
  v1 := arrow(⟨1, 0, 0⟩, shape = arrow, color = red, thickness = 2, shape = cylindrical_arrow) :
  v2 := arrow(⟨−1, 1, 0⟩, shape = arrow, color = blue, thickness = 2, shape = cylindrical_arrow) :
  v3 := arrow(⟨1, −2, 1⟩, shape = arrow, color = green, thickness = 2, shape = cylindrical_arrow) :
  display(e1, e2, e3, v1, v2, v3, axes = boxed, scaling = constrained)
```