Growth Rates for Sequences

Let p, r, a be positive constants

$(\ln n)^p \ll n^r \ll a^n \ll n! \ll n^n$

where $a_n \ll b_n$ means $\lim_{n\to\infty} a_n/b_n = 0$

$$\lim_{n\to\infty} \frac{(\ln n)^p}{n^r} = \frac{\infty}{\infty}$$

L'H $\lim_{n\to\infty} \frac{p(\ln n)^{p-1} \, 1/n}{r n^{r-1}}$

$$\lim_{n\to\infty} \frac{p(\ln n)^{p-1}}{r n^r}$$

$$\lim_{n\to\infty} \frac{3^n}{n^{347}} = \infty$$

I. Series

Given a sequence $\{a_n\}$ we can ask whether we can add all of the numbers together.

i.e. does the expression $\sum_{k=1}^{\infty} a_k$ make sense

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Given a sequence $\{a_n\}_{n=a_0}$, we have another sequence $\{s_n\}_{n=n_0}$ called the sequence of partial sums, given by

$$s_n = \sum_{k=n_0}^{n} a_k$$

If $\lim_{n\to\infty} s_n$ exists, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=n_0}^{n} a_k = \sum_{k=n_0}^{\infty} a_k$

$$\lim s_n = \sum_{k=n_0}^{\infty} a_k$$

Definition $\sum_{k=n_0}^{\infty} a_k$ converges if $\lim_{n\to\infty} s_n$ exists

$$\lim_{n\to\infty} s_n = L \iff \sum_{k=n_0}^{\infty} a_k = L$$

$\sum_{k=n_0}^{\infty} a_k$ diverges if s_n DNE

II Geometric Series

For what values of r does the series $\sum_{k=0} ar^k$ converge.

(Here $\sum_{k=0} a_k$ is geometric if $a_k = ar^k$

We must analyze

$$s_n = a_0 + a_1 + a_2 \ldots a_n$$

$$r \times (s_n = ar^0 + ar^1 + ar^2 \ldots ar^n) \times r$$

$$rs_n = ar + ar^2 + ar^3 \ldots ar^{n+1}$$

$$rs_n = a + ar^{n+1}$$

$$(1-r)\, s_n = a - ar^{n+1}$$

$$s_n = \frac{a - ar^{n+1}}{1-r}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left[\frac{a - ar^{n+1}}{1-r}\right]$$

$$= \frac{1}{1-r} \lim_{n\to\infty}\left[a - ar^{n+1}\right]$$

$$\frac{1}{1-r}\left[a - a\lim_{n\to\infty} r^{n+1}\right]$$

DNE $\Rightarrow |r| > 1,\ r \neq 1$

$$\lim_{n\to\infty} s_n = \begin{cases} \frac{a}{1-r}, & |r|<1 \\ DNE, & |r|>1, r\neq 1 \end{cases}$$

$$\boxed{\sum_{k=0}^{\infty} ar^k = \begin{cases} \frac{a}{1-r}, & |r|<1 \\ \text{diverges}, & |r|>1 \end{cases}}$$

Using the formula

Note $\sum_{k=0}^{\infty} ar^k = a/1-r,\ r<1$

requires 1. The sum start at $k=0$

2. Sum is a constant times a constant to kth power only.

ex $\sum_{k=0}^{\infty} \frac{4^{k+1}}{3^{2k}} =$

$$\frac{4^{k+1}}{3^{2k}} = \frac{4^k 4^1}{(3^2)^k} = \frac{4\cdot 4^k}{9^k}$$

$$4\left(4/9\right)^k$$

So $\sum_{k=0}^{\infty} \frac{4^{k+1}}{3^{2k}} = \sum_{k=0}^{\infty} 4(4/9)^k$

This is now geometric w/ $a=4$ $r=4/9$ It conv since $r<1$

and

$$\sum_{k=0}^{\infty} 4(4/9)^k = \frac{4}{1-4/9}$$