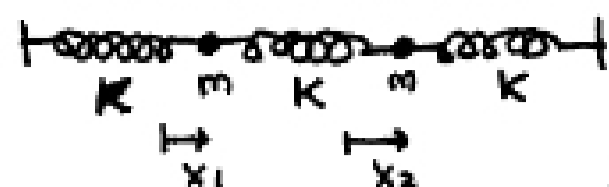


2 mass 3 spring problem: equal masses, equal spring constants

(a: homogeneous problem)



$$m_1 x_1'' = -k_1 x_1 + k_3(x_2 - x_1)$$

$$m_2 x_2'' = -k_2 x_2 + k_3(x_2 - x_1)$$

$$x_1'' = -2\omega^2 x_1 + \omega^2 x_2$$

$$x_2'' = \omega^2 x_1 - 2\omega^2 x_2$$

$$\omega^2 = \frac{K}{m}$$

For simplicity set $\omega = 1$:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = \underbrace{\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\vec{x}'' = A \vec{x}$$

$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda+2)^2 - 1 = \lambda^2 + 4\lambda + 3 = 0, \lambda = -3, -1 \quad (\text{increasing real value ordering})$$

$$= (\lambda+1)(\lambda+3)$$

$$\lambda = -3: A + 3I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{matrix} x_1 + x_2 = 0 \\ x_2 = t \end{matrix} \quad x_1 = -t \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix} \vec{b}_1$$

$$\lambda = -1: A + I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{matrix} x_1 - x_2 = 0 \\ x_2 = t \end{matrix} \quad x_1 = t \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \vec{b}_2$$

$$B = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B^{-1} = \frac{1}{-2} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\vec{x} = B \vec{y} = y_1 \vec{b}_1 + y_2 \vec{b}_2$$

$$\vec{y} = B^{-1} \vec{x}$$

$$A_B = B^{-1} A B = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}, \quad \vec{x}'' = A \vec{x}, \quad B^{-1} (B \vec{y})'' = A B \vec{y} \rightarrow \vec{y}'' = (B^{-1} A B) \vec{y}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \begin{matrix} y_1'' = -3y_1 \\ y_2'' = -y_2 \end{matrix} \quad \begin{matrix} y_1'' + 3y_1 = 0 \\ y_2'' + y_2 = 0 \end{matrix} \xrightarrow{y'' + \omega^2 y = 0} \begin{matrix} y_1 = c_1 \cos \sqrt{3}t + c_2 \sin \sqrt{3}t \\ y_2 = c_3 \cos t + c_4 \sin t \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (c_1 \cos \sqrt{3}t + c_2 \sin \sqrt{3}t) \begin{bmatrix} -1 \\ 1 \end{bmatrix} + (c_3 \cos t + c_4 \sin t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

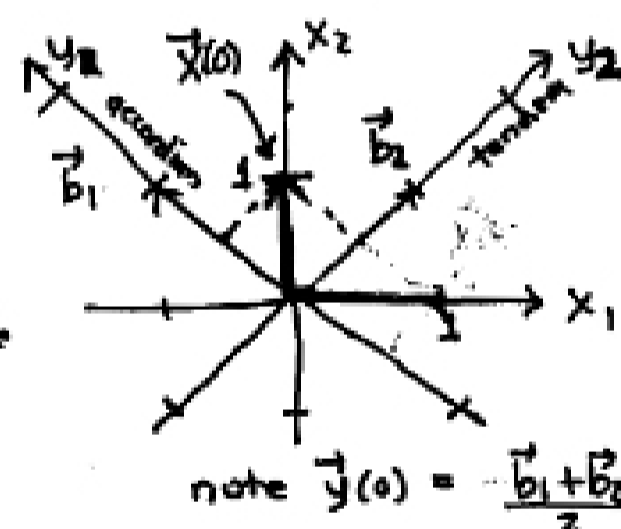
faster accordion mode

slower tandem mode

IVP: $x_1(0) = 0, x_1'(0) = 0$
 $x_2(0) = 1, x_2'(0) = 0$
 (at rest, second mass away from equilibrium)

equal amplitude 180° out of phase

equal amplitude in phase



$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = (-\sqrt{3}c_1 \sin \sqrt{3}t + \sqrt{3}c_2 \cos \sqrt{3}t) \begin{bmatrix} -1 \\ 1 \end{bmatrix} + (-c_3 \sin t + c_4 \cos t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} \rightarrow \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \sqrt{3}c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3}c_2 \\ c_4 \end{bmatrix} \rightarrow \begin{matrix} \sqrt{3}c_2 = 0 \rightarrow c_2 = 0 \\ c_4 = 0 \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{2} \cos \sqrt{3}t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{2} \cos t \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \cos \sqrt{3}t + \frac{1}{2} \cos t \\ \frac{1}{2} \cos \sqrt{3}t + \frac{1}{2} \cos t \end{bmatrix} \quad (\text{beating solutions})$$

$$\omega_b = \frac{\sqrt{3}-1}{2}$$

equal parts of each mode

2 mass 3 spring problem: equal masses, equal spring constants (b: nonhomogeneous problem)

Add force $F_2 = F_{20} \cos 3t$ applied to second mass, set $\frac{F_{20}}{m} = 48$ for simplicity

$$\vec{x}'' = A\vec{x} + \underbrace{\begin{bmatrix} 0 \\ 48 \cos 3t \end{bmatrix}}_{\vec{f}} \quad B^{-1}\vec{f} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 48 \cos 3t \end{bmatrix} = \begin{bmatrix} 24 \cos 3t \\ 24 \cos 3t \end{bmatrix}$$

$$\vec{y}'' = A_B \vec{y} + B^{-1}\vec{f} \quad \begin{aligned} y_1'' &= -3y_1 + 24 \cos 3t & y_1'' + 3y_1 &= 24 \cos 3t \\ y_2'' &= -y_2 + 24 \cos 3t & y_1'' + y_2 &= 24 \cos 3t \end{aligned}$$

method of undetermined coefficients: $y_{1p} = C_5 \cos 3t \rightarrow (-9+3)C_5 \cos 3t = 24 \cos 3t \rightarrow C_5 = -4$
 $y_{2p} = C_6 \cos 3t \rightarrow (-9+1)C_6 \cos 3t = 24 \cos 3t \rightarrow C_6 = -3$

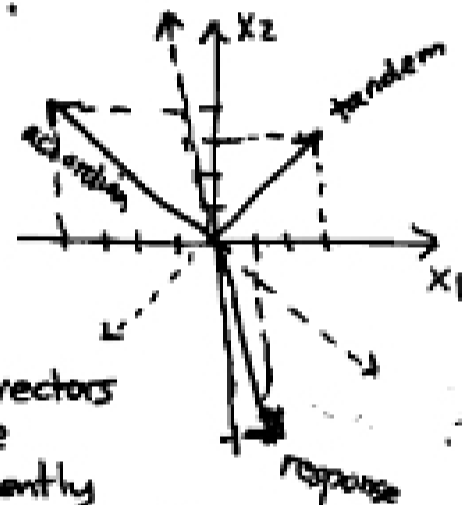
$$\vec{x}_p = B\vec{y}_p = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ -3 \end{bmatrix} \cos 3t = \begin{bmatrix} 1 \\ -7 \end{bmatrix} \cos 3t$$

$$\vec{x} = \vec{x}_h + \vec{x}_p = \underbrace{\begin{pmatrix} c_1 \cos \sqrt{3}t \\ + c_2 \sin \sqrt{3}t \end{pmatrix}}_{\text{accordion}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \underbrace{\begin{pmatrix} c_3 \cos t \\ + c_4 \sin t \end{pmatrix}}_{\text{tandem}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underbrace{\cos 3t}_{\text{response}} \begin{bmatrix} 1 \\ -7 \end{bmatrix} \quad \text{general solutions}$$

IVP at rest, at equilibrium: again $c_2 = 0 = c_4$ from $\vec{x}'(0) = 0$.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{pmatrix} c_1 \\ c_3 \end{pmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{pmatrix} c_3 - 3 \\ c_3 - 3 \end{pmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 - 4 \\ c_3 - 3 \end{bmatrix} \rightarrow \begin{aligned} c_1 &= 4 \\ c_3 &= 3 \end{aligned}$$

$$\vec{x} = 4 \cos \sqrt{3}t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 3 \cos t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cos 3t \begin{bmatrix} 1 \\ -7 \end{bmatrix}$$



These 3 vectors oscillate independently
 no common period
 since frequencies 1, $\sqrt{3}$, 3
 have some irrational ratios.

(c: resonance) replace frequency 3 by ω , repeat:

$$(-\omega^2 + 3)C_5 \cos \omega t = 24 \cos \omega t \rightarrow C_5 = 24 / (3 - \omega^2)$$

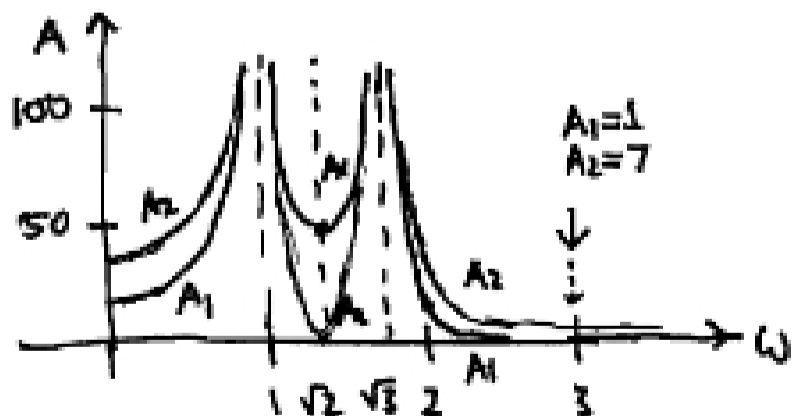
$$(-\omega^2 + 1)C_6 \cos \omega t = 24 \cos \omega t \rightarrow C_6 = 24 / (1 - \omega^2)$$

$$\vec{y}_p = 24 \cos \omega t \begin{bmatrix} \frac{1}{3 - \omega^2} \\ \frac{1}{1 - \omega^2} \end{bmatrix}$$

$$\vec{x}_p = B\vec{y}_p = 24 \cos \omega t \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3 - \omega^2} \\ \frac{1}{1 - \omega^2} \end{bmatrix} = 24 \cos \omega t \begin{bmatrix} -\frac{1}{3 - \omega^2} + \frac{1}{1 - \omega^2} \\ \frac{1}{3 - \omega^2} + \frac{1}{1 - \omega^2} \end{bmatrix} = \frac{48}{(-\omega^2)(3 - \omega^2)} \begin{bmatrix} 1 - \omega^2 \\ 2 - \omega^2 \end{bmatrix} \cos \omega t$$

$$\vec{A} = \langle A_1, A_2 \rangle = \frac{48}{|1 - \omega^2| |3 - \omega^2|} \langle 1, |2 - \omega^2| \rangle$$

amplitude (with sign) vector gives direction of oscillation in $x_1 - x_2$ plane. $= \begin{bmatrix} \pm A_1 \\ \pm A_2 \end{bmatrix}$ amplitudes up to sign



If we drive the system with a frequency much closer to either natural frequency 1 or $\sqrt{3}$ we would get a much bigger response.

Example: $\vec{A}(1.1) = \langle 127.7, 100.9 \rangle$ compared to:
 $\vec{A}(3) = \langle 1, 7 \rangle$

d) IVP exercise: impose above initial conditions to determine c_1, c_3 . These will be correspondingly larger near resonant frequencies.