

Lecture L21 - 2D Rigid Body Dynamics

Introduction

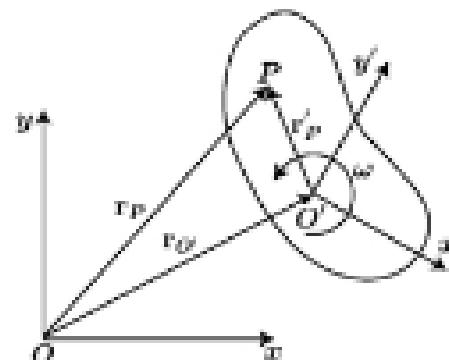
In lecture 11, we derived conservation laws for angular momentum of a system of particles, both about the center of mass, point G , and about a fixed (or at least non-accelerating) point O . We then extended this derivation to the motion of a rigid body in two-dimensional plane motion including both translation and rotation. We obtained statements about the conservation of angular momentum about both a fixed point and about the center of mass. Both are powerful statements. However, each has its own separate requirements for application. In the case of motion about a fixed point, the point must have **zero acceleration**. Thus the instantaneous center of rotation, for example the point of contact of a cylinder rolling on a plane, cannot be used as the origin of our coordinates. For motion about the center of mass, no such restriction applies and we may obtain the statement of conservation of angular momentum about the center of mass even if this point is accelerating.

Kinematics of Two-Dimensional Rigid Body Motion

Even though a rigid body is composed of an infinite number of particles, the motion of these particles is constrained to be such that the body remains a rigid body during the motion. In particular, the only degrees of freedom of a 2D rigid body are *translation* and *rotation*.

Parallel Axes

Consider a 2D rigid body which is rotating with angular velocity ω about point O' , and, simultaneously, point O' is moving relative to a fixed reference frame x and y with origin O .



In order to determine the motion of a point P in the body, we consider a second set of axes $x'y'$, always parallel to xy , with origin at O' , and write,

$$\mathbf{r}_P = \mathbf{r}_{O'} + \mathbf{r}'_P \quad (1)$$

$$\mathbf{v}_P = \mathbf{v}_{O'} + (\mathbf{v}_P)_{O'} \quad (2)$$

$$\mathbf{a}_P = \mathbf{a}_{O'} + (\mathbf{a}_P)_{O'} . \quad (3)$$

Here, \mathbf{r}_P , \mathbf{v}_P and \mathbf{a}_P are the position, velocity and acceleration vectors of point P , as observed by O ; $\mathbf{r}_{O'}$ is the position vector of point O' ; and \mathbf{r}'_P , $(\mathbf{v}_P)_{O'}$ and $(\mathbf{a}_P)_{O'}$ are the position, velocity and acceleration vectors of point P , as observed by O' . Relative to point O' , all the points in the body describe a circular orbit ($r'_P = \text{constant}$), and hence we can easily calculate the velocity,

$$(\mathbf{v}_P)_{O'} = r'_P \dot{\theta} = r\omega ,$$

or, in vector form,

$$(\mathbf{v}_P)_{O'} = \boldsymbol{\omega} \times \mathbf{r}'_P ,$$

where $\boldsymbol{\omega}$ is the angular velocity vector. The acceleration has a circumferential and a radial component,

$$((\mathbf{a}_P)_{O'})_{\theta} = r'_P \ddot{\theta} = r'_P \dot{\omega}, \quad ((\mathbf{a}_P)_{O'})_r = -r'_P \dot{\theta}^2 = -r'_P \omega^2 .$$

Noting that $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$ are perpendicular to the plane of motion (i.e. $\boldsymbol{\omega}$ can change magnitude but not direction), we can write an expression for the acceleration vector as,

$$(\mathbf{a}_P)_{O'} = \dot{\boldsymbol{\omega}} \times \mathbf{r}'_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_P) .$$

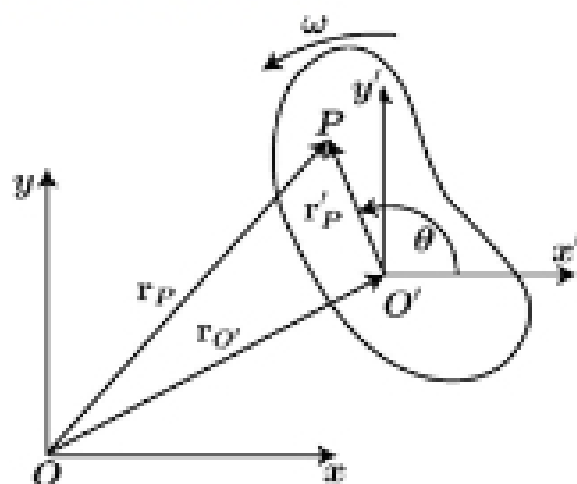
Recall here that for any three vectors \mathbf{A} , \mathbf{B} and \mathbf{C} , we have $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$. Therefore $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_P) = (\boldsymbol{\omega} \cdot \mathbf{r}'_P)\boldsymbol{\omega} - \omega^2 \mathbf{r}'_P = -\omega^2 \mathbf{r}'_P$. Finally, equations 2 and 3 become,

$$\mathbf{v}_P = \mathbf{v}_{O'} + \boldsymbol{\omega} \times \mathbf{r}'_P \quad (4)$$

$$\mathbf{a}_P = \mathbf{a}_{O'} + \dot{\boldsymbol{\omega}} \times \mathbf{r}'_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_P) . \quad (5)$$

Body Axes

An alternative description can be obtained using body axes. Now, let $x'y'$ be a set of axes which are **rigidly attached** to the body and have the origin at point O' .



Then, the motion of an arbitrary point P can be expressed in terms of the general expressions for relative motion. Recall that,

$$\mathbf{r}_P = \mathbf{r}_{O'} + \mathbf{r}'_P \quad (6)$$

$$\mathbf{v}_P = \mathbf{v}_{O'} + (\mathbf{v}_P)_{O'} + \boldsymbol{\omega} \times \mathbf{r}'_P \quad (7)$$

$$\mathbf{a}_P = \mathbf{a}_{O'} + (\mathbf{a}_P)_{O'} + 2\boldsymbol{\omega} \times (\mathbf{v}_P)_{O'} + \dot{\boldsymbol{\omega}} \times \mathbf{r}'_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_P) . \quad (8)$$

Here, \mathbf{r}_P , \mathbf{v}_P and \mathbf{a}_P are the position, velocity and acceleration vectors of point P as observed by O ; $\mathbf{r}_{O'}$ is the position vector of point O' ; \mathbf{r}'_P , $(\mathbf{v}_P)_{O'}$ and $(\mathbf{a}_P)_{O'}$ are the position, velocity and acceleration vectors of point P as observed by O' ; and $\boldsymbol{\Omega} = \boldsymbol{\omega}$ and $\dot{\boldsymbol{\Omega}} = \dot{\boldsymbol{\omega}}$ are the body angular velocity and acceleration.

Since we only consider 2D motions, the angular velocity vector, $\boldsymbol{\Omega}$, and the angular acceleration vector, $\dot{\boldsymbol{\Omega}}$, do not change direction. Furthermore, because the body is rigid, the relative velocity $(\vec{v}_P)_{O'}$ and acceleration $(\vec{a}_P)_{O'}$ of any point in the body, as observed by the body axes, is zero. Thus, equations 7 and 8 simplify to,

$$\mathbf{v}_P = \mathbf{v}_{O'} + \boldsymbol{\omega} \times \mathbf{r}'_P \quad (9)$$

$$\mathbf{a}_P = \mathbf{a}_{O'} + \dot{\boldsymbol{\omega}} \times \mathbf{r}'_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'_P) , \quad (10)$$

which are identical to equations 4 and 5, as expected. Note that their vector forms are equal. If at $t=0$, the frame x' , y' (and eventually z') are instantaneously aligned with the frame x , y , the components of the vectors are equal. If not, then a coordinate transformation is required.

Invariance of $\boldsymbol{\omega}$ and $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$

The angular velocity, $\boldsymbol{\omega}$, and the angular acceleration, $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$, are invariant with respect to the choice of the reference point O' . In other words, this means that an observer using parallel axes situated *anywhere* in the rigid body will observe all the other points of the body turning around, in circular paths, with the same angular velocity and acceleration. Mathematically, this can be seen by considering an arbitrary point in the body O'' and writing,

$$\mathbf{r}'_P = \mathbf{r}'_{O''} + \mathbf{r}''_P .$$

