Central Force Motion – Getting Started

Center of mass $\vec{R} \equiv \frac{1}{M}\sum_i m_i \vec{r}_i$, $M \equiv \sum_i m_i$

In C.M. system, $\vec{r}_i{}' \equiv \vec{r}_i - \vec{R}$

$$\frac{1}{M}\sum_i m_i \vec{r}_i{}' = \frac{1}{M}\sum_i m_i(\vec{r}_i - \vec{R}_i) = \frac{1}{M}\sum_i m_i \vec{r}_i - \vec{R} = 0$$

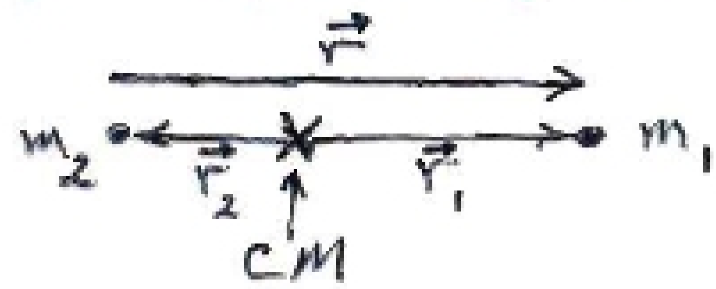
$$\sum_i m_i \vec{r}_i{}' = 0 \quad \text{(center of mass)}$$

Now we switch to CM system and use $\vec{r}_i$ instead of $\vec{r}_i{}'$

Also, we consider just two objects: m_1 and m_2

Therefore:
Define:
$$\begin{cases} m_1\vec{r}_1 + m_2\vec{r}_2 = 0 \\ \vec{r}_1 - \vec{r}_2 \equiv \vec{r} \end{cases}$$

Separation Vector



Solution: $\vec{r}_1 = \frac{m_2}{m_1+m_2}\vec{r}$, $\vec{r}_2 = \frac{-m_1}{m_1+m_2}\vec{r}$

$$\mu \equiv \frac{m_1 m_2}{m_1+m_2} \qquad \vec{r}_1 = \frac{\mu}{m_1}\vec{r} \qquad \vec{r}_2 = \frac{-\mu}{m_2}\vec{r}$$

$$T = \frac{1}{2}m_1|\dot{\vec{r}}_1|^2 + \frac{1}{2}m_2|\dot{\vec{r}}_2|^2 = \frac{1}{2}\mu^2|\dot{\vec{r}}|^2\left(\frac{1}{m_1}+\frac{1}{m_2}\right)$$

$$\boxed{T = \frac{1}{2}\mu|\dot{\vec{r}}|^2} \qquad \text{since } \frac{1}{m_1}+\frac{1}{m_2} = \frac{1}{\mu}$$

We use polar coordinates (r, θ) to describe the separation vector $\vec{r}$.

After finding the motion $(r(t), \theta(t))$ we have also the motions of the two bodies (in the center-of-mass reference frame):

$$\vec{r}_1(t) = \frac{\mu}{m_1}\,\vec{r}(t) \Leftrightarrow r_1(t) = \frac{\mu}{m_1}\,r(t),\ \theta_1(t) = \theta(t)$$
$$\vec{r}_2(t) = \frac{-\mu}{m_2}\,\vec{r}(t) \Leftrightarrow r_2(t) = +\frac{\mu}{m_2}\,r(t),\ \theta_2(t) = \theta(t) + \pi$$

$$L = T - U = \tfrac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r)$$

$$\frac{\partial L}{\partial \theta} = 0 \Rightarrow \underline{\ell} \equiv \frac{\partial L}{\partial \dot{\theta}} = \underline{\mu r^2 \dot{\theta} \text{ is constant}}$$

Constant "areal velocity"

$$\frac{dA}{dt} = \frac{1}{2}\,r(r\dot{\theta}) = \frac{\ell}{2\mu} \quad \text{[See fig 8-3]}$$

"Radius vector sweeps out area $\frac{\ell}{2\mu}$ per unit time"

Kepler's second law.
True for <u>any</u> central force, not just for gravitation.

Conservative system $\Rightarrow$ $E \equiv T + U = \text{constant}$

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r) \quad \text{constant}$$

$$\frac{1}{2}\mu\dot{r}^2 = E - U - \frac{\ell^2}{2\mu r^2}$$

$$\boxed{\dot{r} = \pm\sqrt{\frac{2}{\mu}\left(E - U - \frac{\ell^2}{2\mu r^2}\right)}}$$

$$dt = \pm\frac{dr}{\sqrt{\frac{2}{\mu}\left(E - U - \frac{\ell^2}{2\mu r^2}\right)}} \quad \Rightarrow \quad r(t)$$

Find, instead, $\theta(r)$ or $r(\theta)$

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr$$

Use $\dot{\theta} = \frac{\ell}{\mu r^2}$ and $\dot{r} = \pm\sqrt{\frac{2}{\mu}\left(E - U - \frac{\ell^2}{2\mu r^2}\right)}$

$$\theta(r) = \int \frac{\pm(\ell/r^2)\,dr}{\sqrt{2\mu\left(E - U - \frac{\ell^2}{2\mu r^2}\right)}}$$