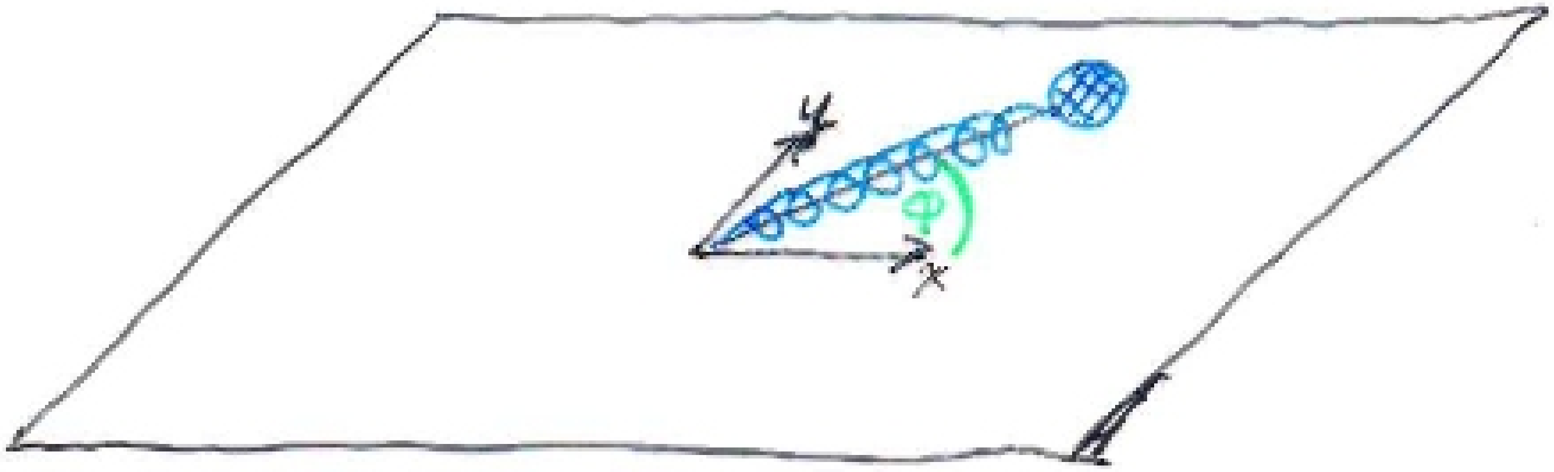
Example problem: Pivoted spring with mass

Frictionless horizontal plane

Spring constant k, unstretched length b



$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2)$$

$$U = \frac{1}{2} k (r - b)^2$$

$$L = T - U = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2) - \frac{1}{2} k (r-b)^2$$

$$\frac{\partial L}{\partial \varphi} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\varphi}} = \text{const} \Rightarrow \boxed{m r^2 \dot{\varphi} = Q \ (\text{constant})}$$

(angular momentum conserved)

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 \Rightarrow \frac{d}{dt}(m\dot{r}) - m r \dot{\varphi}^2 + k(r-b) = 0$$

$$\boxed{m\ddot{r} = \frac{Q^2}{m r^3} - k(r-b)}$$

Conserved Quantities

If L does <u>not</u> depend on some generalized coordinate q_i, then $\frac{\partial L}{\partial \dot{q}_i}$ is conserved.

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \cancel{\frac{\partial L}{\partial q_i}} \Leftrightarrow \frac{\partial L}{\partial \dot{q}_i} \text{ is constant in time.}$$

Example: $\frac{\partial L}{\partial z} = 0$ (No dependence of the Lagrangian on one Cartesian coordinate)

$\Leftrightarrow \frac{\partial L}{\partial \dot{z}} = \text{const} \Leftrightarrow m\dot{z} = \text{constant}$

(momentum conservation in the z-direction)

<u>Example</u>: $\frac{\partial L}{\partial \theta} = 0$ (No dependence on an angular coordinate)

$\Leftrightarrow \frac{\partial L}{\partial \dot{\theta}} = \text{const} \Leftrightarrow mr^2\dot{\theta} = \text{const.}$

(angular momentum conservation)

$\frac{\partial L}{\partial \dot{q}_i}$ is the "momentum conjugate to the coordinate q_i"

Energy is a special case, conjugate to t (in a sense).

Second form of Euler-Lagrange equation

[See section 6.4]

$$L(q_i, \dot{q}_i; t)$$

$$\frac{d}{dt} L \equiv \sum_i \frac{\partial L}{\partial q_i} \dot{q}_i + \sum \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i + \frac{\partial L}{\partial t}$$

$$\begin{aligned}
\frac{d}{dt} \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} &= \sum_i \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i} + \sum \dot{q}_i \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \\
&= \frac{dL}{dt} - \frac{\partial L}{\partial t} - \sum_i \frac{\partial L}{\partial q_i} \dot{q}_i + \sum \dot{q}_i \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \\
&= \frac{dL}{dt} - \frac{\partial L}{\partial t} - \sum_i \left(\cancel{\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}} \right) \dot{q}_i
\end{aligned}$$

$$\frac{\partial L}{\partial t} - \frac{d}{dt}\left(L - \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) = 0$$

Whenever $\frac{\partial L}{\partial t} = 0$, then

$$\begin{aligned}
L - \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} &= \text{constant of motion} \\
&\equiv -H
\end{aligned}$$

[The Hamiltonian functional is <u>constant</u> in time if $\frac{\partial L}{\partial t} = 0$]

$H = E = T + U$ <u>if</u> $q_i(x_j)$ (Transformation from Cartesian coordinates <u>not</u> depending on time.)