Newtonian Gravitation

$$\vec{F} = -G\frac{mM}{r^2}\,\vec{e}_r$$

Force on m due to M.
$\vec{e}_r$ is unit vector pointing from M to m.
$r^2 \equiv |\vec{r}_m - \vec{r}_M|^2$

Add forces linearly (due to different masses).

(Force on m) $\vec{F} = -Gm\int_V \frac{\rho(r)\,\vec{e}_r}{r^2}\,dv$
$\vec{e}_r = (\vec{r}_m - \vec{r})/|\vec{r}_m - \vec{r}|$
$r^2 = |\vec{r}_m - \vec{r}|^2$

Gravitational field $\vec{g}$

$$\vec{g} \equiv \frac{\vec{F}}{m} = \begin{cases} -G\frac{M}{r^2}\,\vec{e}_r & \text{Point mass } M \\ -G\int_V \frac{\rho(r)\,\vec{e}_r}{r^2}\,dv & \text{Distributed mass} \end{cases}$$

Gravitational potential Φ

Potential energy per unit mass

$$-\vec{\nabla}\Phi = \vec{g}\,, \qquad \Phi = -G\frac{M}{r} \quad \text{(for point mass)}$$

Note that $\int_S \vec{g}\cdot d\vec{a} = -4\pi G M$

for any surface S enclosing (total) mass M.

Generally: $\Phi(\vec{r}) = -G\int_V \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\,dv'$

- $\int_S \vec{g} \cdot d\vec{a} = -4\pi G M$ $\quad$ M = enclosed mass
- $\int_V \rho \, dv = M$
- $\int_V \vec{\nabla} \cdot \vec{g} \, dv = \int_S \vec{g} \cdot d\vec{a}$ $\quad$ (Gauss' Law)

$\int_V \vec{\nabla} \cdot \vec{g} \, dv = \int_V (-4\pi G \rho) \, dv$ for <u>any</u> V

$$\Rightarrow \quad \vec{\nabla} \cdot \vec{g} = -4\pi G \rho$$

$\vec{g} = -\vec{\nabla} \Phi \Rightarrow \quad \nabla^2 \Phi = +4\pi G \rho$ (Poisson)

Wherever $\rho = 0$, (outside bodies) $\quad \nabla^2 \Phi = 0$ (Laplace)

Uniform spherical shell

- No gravitational force anywhere inside.
- Anywhere outside the force is the same as that of a point source at the shell's center with the total mass of the shell.

$$\Phi = -G\int_S \frac{\rho}{r}\, da = -\int_0^{\pi} G\,\frac{\rho\,(2\pi a \sin\theta)\,a\,d\theta}{(R^2+a^2-2aR\cos\theta)^{1/2}}$$

$$= -\frac{GM}{2aR}\left[(R+a)-|R-a|\right]$$

$$= \begin{cases} -\dfrac{GM}{R} & R>a \quad (\Leftrightarrow \text{Point source}) \\ -\dfrac{GM}{a} & R<a \quad (\text{No force}) \end{cases}$$