Resonance

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos(\omega t)$$

$$x_p(t) = D\cos(\omega t - \delta)$$

$$D = \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}} \qquad \delta = \tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right)$$

$$\frac{dD}{d\omega} = 0 \Rightarrow \omega \equiv \omega_R = \sqrt{\omega_0^2 - 2\beta^2}$$

Quality factor $Q \equiv \dfrac{\omega_R}{2\beta}$

$$\left[Q \equiv 2\pi \times \frac{\text{Peak Energy Stored}}{\text{Energy Dissipated per Cycle}} = \omega \frac{\text{Energy Stored}}{\text{Power Loss}}\right]$$

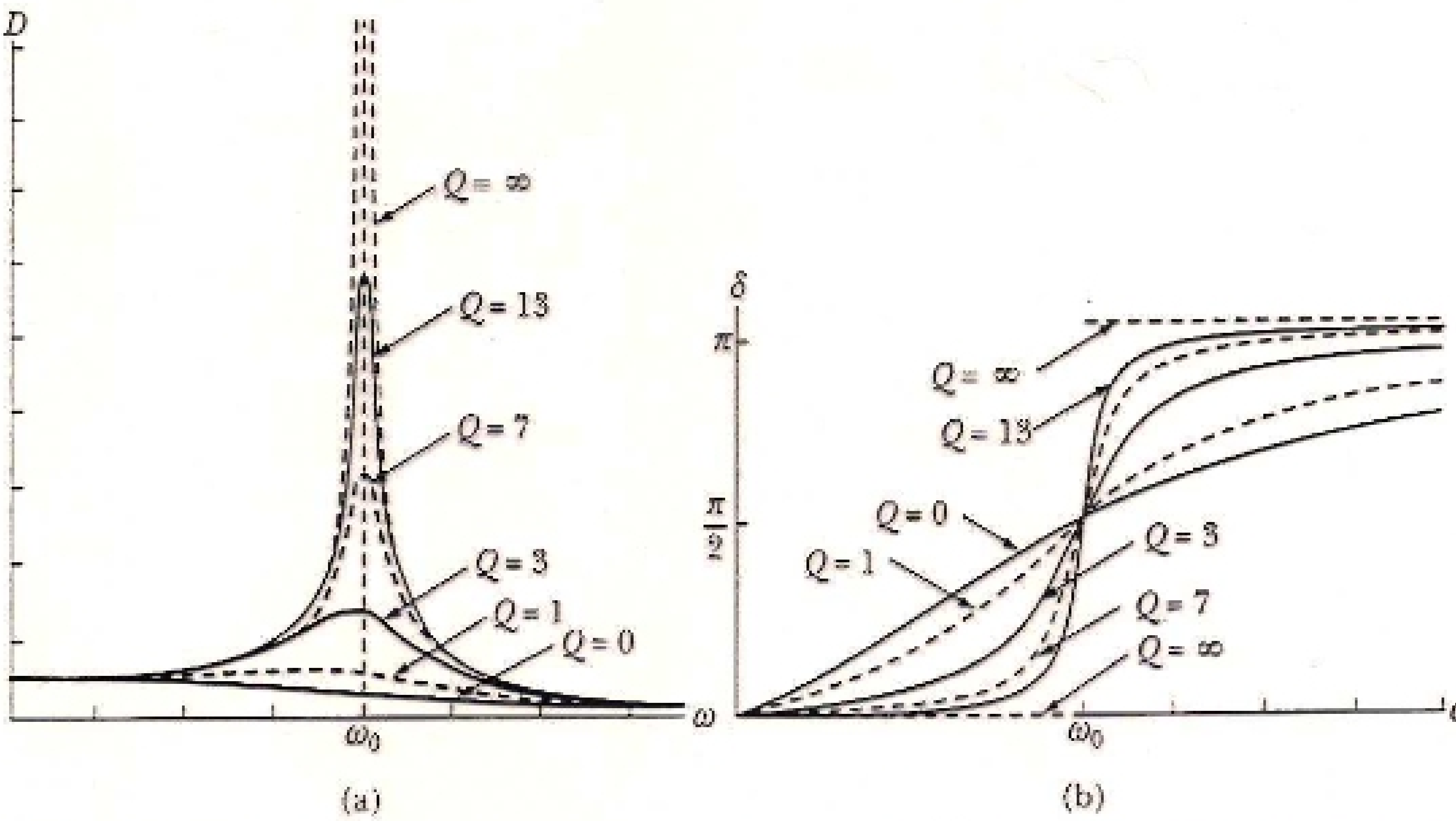

FIGURE 3-16 (a) The amplitude D is displayed as a function of the driving frequency ω for various values of the quality factor Q. Also shown is (b) the phase angle δ, which is the phase angle between the driving force and the resultant motion.

Driven damped oscillator

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{1}{m}F(t)$$

$$\frac{1}{m}F(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega t + b_n \sin n\omega t\right)$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \alpha_n \cos(n\omega t - \phi_n)$$

For each n, we have the solution

$$x_n(t) = \frac{\alpha_n}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + 4n^2\omega^2\beta^2}} \cos(n\omega t - \phi_n - \delta_n)$$

$$\delta_n \equiv \tan^{-1}\frac{2n\omega\beta}{\omega_0^2 - n^2\omega^2}$$

$$x(t) = \sum_n x_n(t)$$ Explicit series solution

Note: The solution for superposition of forces is the superposition of separate solutions because our differential operator is <u>linear</u>:

$$Lx = \frac{F(t)}{m} \quad \text{with} \quad L = \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2.$$

$$\left.\begin{aligned} Lx_1 &= \tfrac{1}{m}F_1 \\ Lx_2 &= \tfrac{1}{m}F_2 \end{aligned}\right\} \Rightarrow L(x_1 + x_2) = Lx_1 + Lx_2 = \tfrac{1}{m}F_1 + \tfrac{1}{m}F_2$$