Fourier Series

If $F(\theta)$ is piecewise continuous and periodic (i.e. $F(\theta+2\pi) = F(\theta)$ for all θ) [and $F(\theta)$ is "normalized" to its "average value" at discontinuities], then its Fourier series converges to $F(\theta)$ at every point θ.

$$F(\theta) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\theta + b_n \sin n\theta\right)$$

where
$$\begin{cases} a_n \equiv \frac{1}{\pi}\int_{-\pi}^{\pi} F(\theta)\cos n\theta \, d\theta \\ b_n \equiv \frac{1}{\pi}\int_{-\pi}^{\pi} F(\theta)\sin n\theta \, d\theta \end{cases}$$

Set $\theta = \frac{2\pi}{\tau} t = \omega t$ and $f(t) = F(\omega t)$:

Note: $f(t+\tau) = f(t)$, period τ

$$f(t) = F(\omega t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega t + b_n \sin n\omega t\right)$$

$$\begin{cases} a_n = \frac{\omega}{\pi}\int_{-\tau/2}^{\tau/2} f(t)\cos n\omega t \, dt \\ b_n = \frac{\omega}{\pi}\int_{-\tau/2}^{\tau/2} f(t)\sin n\omega t \, dt \end{cases}$$

Any function on a domain of length τ can be extended as a periodic function of period τ, so this expansion is general.

Fourier coefficient verification

Note:

$$\sin(m\theta - n\theta) = \sin m\theta \cos n\theta - \cos m\theta \sin n\theta$$
$$\cos(m\theta - n\theta) = \cos m\theta \cos n\theta + \sin m\theta \sin n\theta$$
$$\sin(m\theta + n\theta) = \sin m\theta \cos n\theta + \cos m\theta \sin n\theta$$
$$\cos(m\theta + n\theta) = \cos m\theta \cos n\theta - \sin m\theta \sin n\theta$$

$$\int_{-\pi}^{\pi} \sin m\theta \cos n\theta \, d\theta = \frac{1}{2}\int_{-\pi}^{\pi}\left[\sin(m-n)\theta + \sin(m+n)\theta\right] d\theta = 0$$

$$\int_{-\pi}^{\pi} \cos m\theta \cos n\theta \, d\theta = \frac{1}{2}\int_{-\pi}^{\pi}\left[\cos(m-n)\theta + \cos(m+n)\theta\right] d\theta = \pi\delta_{mn}$$

$$\int_{-\pi}^{\pi} \sin m\theta \sin n\theta \, d\theta = \frac{1}{2}\int_{-\pi}^{\pi}\left[\cos(m-n)\theta - \cos(m+n)\theta\right] d\theta = \pi\delta_{mn}$$

$$\int_{-\pi}^{\pi} \sin m\theta \cos n\theta \, d\theta = 0$$

$$\int_{-\pi}^{\pi} \sin m\theta \sin n\theta \, d\theta = \pi\delta_{mn} = \int_{-\pi}^{\pi} \cos m\theta \cos n\theta \, d\theta$$

$$F(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\theta + b_n \sin n\theta\right)$$

$$\Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} F(\theta) \cos m\theta \, d\theta = a_m$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} F(\theta) \sin m\theta \, d\theta = b_m$$