Math 2210-3
Monday March 1
§12.5 Directional derivatives and the gradient

Recall: $f(x,y)$ differentiable at (x_0,y_0) means it has a good tangent approximation there:

$$f(x_0+h_1, y_0+h_2) = f(x_0,y_0) + h_1 f_x(x_0,y_0) + h_2 f_y(x_0,y_0) + h_1\varepsilon_1(\vec{h}) + h_2\varepsilon_2(\vec{h}) \qquad , \quad \varepsilon_1,\varepsilon_2 \to 0 \text{ as } \vec{h}\to\vec{0}$$

i.e.

$$f(x,y) = \underbrace{f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}_{T(x,y)} + \vec{h}\cdot\vec{\varepsilon}(\vec{h}) \qquad \vec{\varepsilon}(\vec{h})\to\vec{0} \text{ as } \vec{h}\to\vec{0}$$

and if we write this in vector notation, we see that the definition makes sense for a fcn of 3, 4, or n variables:

$$f(\vec{p}_0+\vec{h}) = f(\vec{p}_0) + \nabla f(\vec{p}_0)\cdot\vec{h} + \vec{h}\cdot\vec{\varepsilon}(\vec{h})$$

$\vec{\varepsilon}(\vec{h})\to\vec{0}$ as $\vec{h}\to\vec{0}$.

$\nabla f(\vec{p}_0)$ = the vector of partial derivs of f, at $\vec{p}_0$
called gradient of f at $\vec{p}_0$

Theorem
- If the partial derivs of f are continuous near $\vec{p}_0$, then f is differentiable at $\vec{p}_0$
- Theorem: f differentiable at $\vec{p}_0$ implies f continuous there. Converse not true

Reason

$$f(\vec{p}_0+\vec{h}) - f(\vec{p}_0) = \nabla f(p_0)\cdot\vec{h} + \vec{h}\cdot\vec{\varepsilon}(\vec{h})$$

$$\Rightarrow \quad |f(\vec{p}_0+\vec{h}) - f(\vec{p}_0)| \le |\qquad\quad|$$

$$\le |\quad| + |\quad| = \|\nabla f(p_0)\|\|\vec{h}\||\cos\theta_1| + \|\vec{h}\|\|\vec{\varepsilon}(\vec{h})\||\cos\theta_2|$$

$$\le \|\nabla f(p_0)\|\|\vec{h}\| + \|\vec{h}\|\|\vec{\varepsilon}(\vec{h})\|$$

$$\to 0 \text{ as } \vec{h}\to\vec{0}.$$

Directional derivatives : Recall

$$f_x(x_0,y_0) = \lim_{\substack{h\to 0 \\ \uparrow \text{scalar}}} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} = \text{rate of change of } f \text{ in } \hat{\imath} \text{ direction}$$

$$f_y(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0,y_0+h)-f(x_0,y_0)}{h} = \text{rate of change of } f \text{ in } \hat{\jmath} \text{ direction.}$$

Now,
Let $\vec{u}$ be a unit vector in any direction, f a function of several variables, differentiable at $\vec{p}_0$.

$$\boxed{D_{\vec{u}} f(\vec{p}_0) := \lim_{\substack{h\to 0 \\ \uparrow \\ \text{scalar}}} \frac{f(\vec{p}_0+h\vec{u})-f(\vec{p}_0)}{h}}$$

is called the directional derivative of f in direction $\vec{u}$, at $\vec{p}_0$ and generalizes the concept of partial derivative, to any direction

Examples : this includes partial derivatives as a special case :

$D_{\hat{\imath}} f(x_0,y_0) =$

$D_{\hat{k}} f(x_0,y_0,z_0) =$

$D_{\langle 0,0,0,1\rangle} f(x_0,y_0,z_0,w_0) =$

Theorem : Directional derivatives are easy to compute, if f is differentiable at at $\vec{p}_0$.

proof : Use the tangent approximation formula

$$f(\vec{p}_0+h\vec{u}) = f(\vec{p}_0) + \nabla f(\vec{p}_0)\cdot h\vec{u} + h\vec{u}\cdot\vec{\varepsilon}(h\vec{u})$$

$$\frac{f(p_0+h\vec{u})-f(\vec{p}_0)}{h} = \frac{\nabla f(\vec{p}_0)\cdot \cancel{h}\vec{u}}{\cancel{h}} + \frac{\cancel{h}\vec{u}\cdot\vec{\varepsilon}(h\vec{u})}{\cancel{h}}$$

take $\lim_{h\to 0}$: $\boxed{D_{\vec{u}} f(\vec{p}_0) = \nabla f(\vec{p}_0)\cdot\vec{u}}$

Because $\vec{u}\cdot\vec{\varepsilon}(h\vec{u}) \le \|\vec{u}\|\|\vec{\varepsilon}\| = \|\vec{\varepsilon}\|$

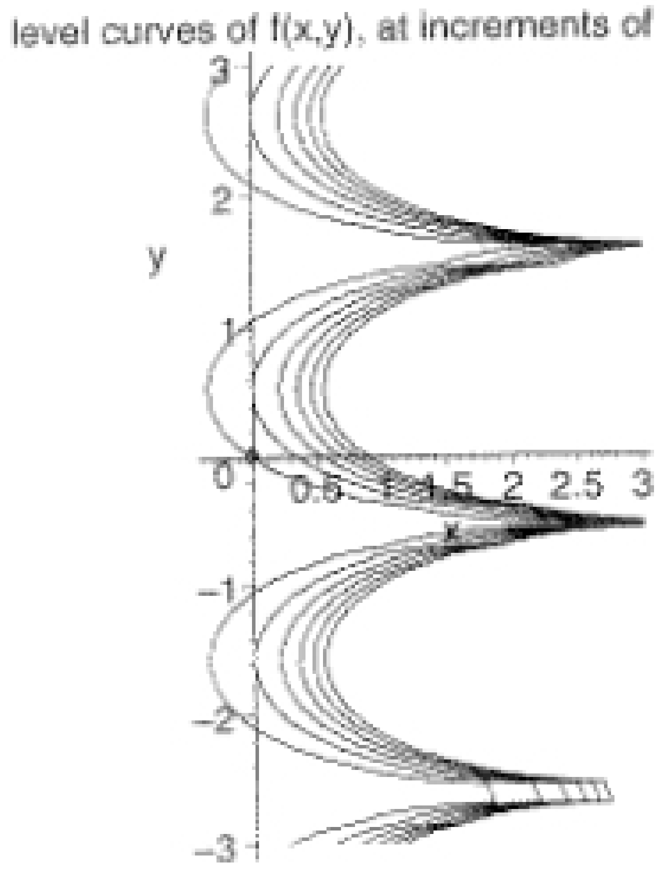
$\ge -\|\vec{u}\|\|\vec{\varepsilon}\| = -\|\vec{\varepsilon}\| \to 0$, $\to 0$, $\to 0$ as $h\to 0$

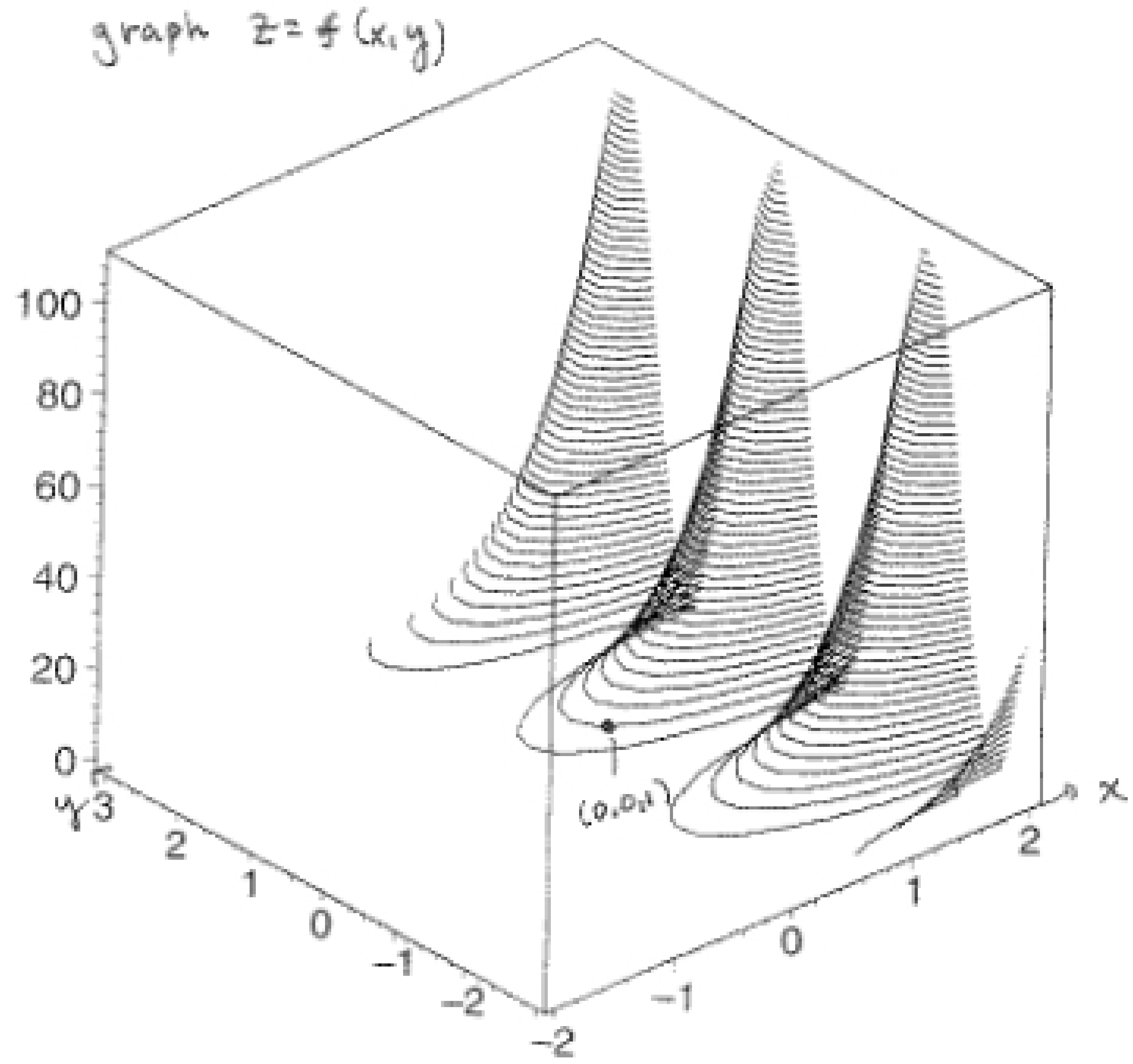
More fun with index cards

$$f(x,y) = e^{2x}(1+\sin 3y)$$

$$(x_0,y_0) = (0,0).$$

Far view





Medium view