Lecture #19 More About MHD Waves

I. Review

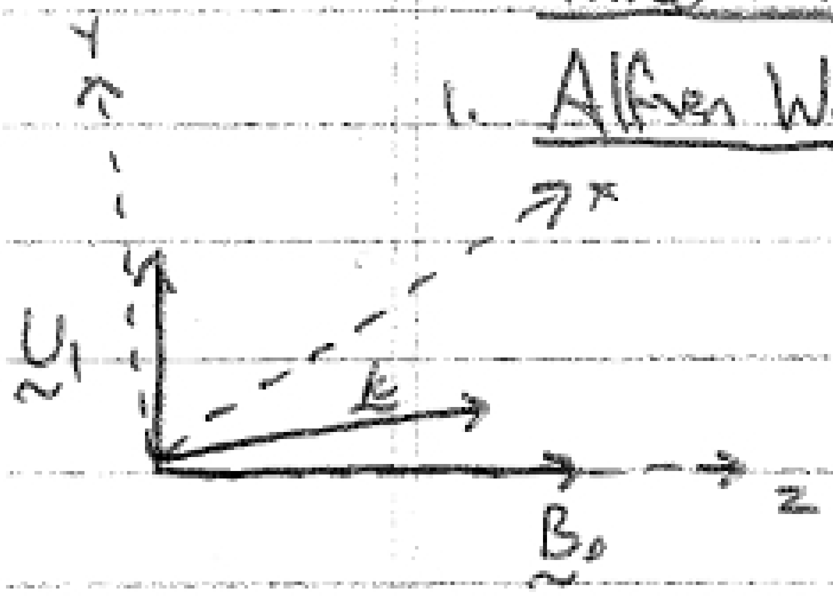
A. Last time, we linearized the MHD equations, assumed plane wave (Fourier) Solutions, and solved to obtain the MHD Dispersion Relation:

$$(\omega^2 - k^2\cos^2\theta\, V_A^2)\left[\omega^4 - \omega^2 k^2 (C_S^2+V_A^2) + k^4\cos^2\theta\, C_S^2 V_A^2\right] = 0$$

where 

$\underset{\sim}{B_0}\cdot\underset{\sim}{k} = B_0 k\cos\theta$

B. Three Wave Modes:

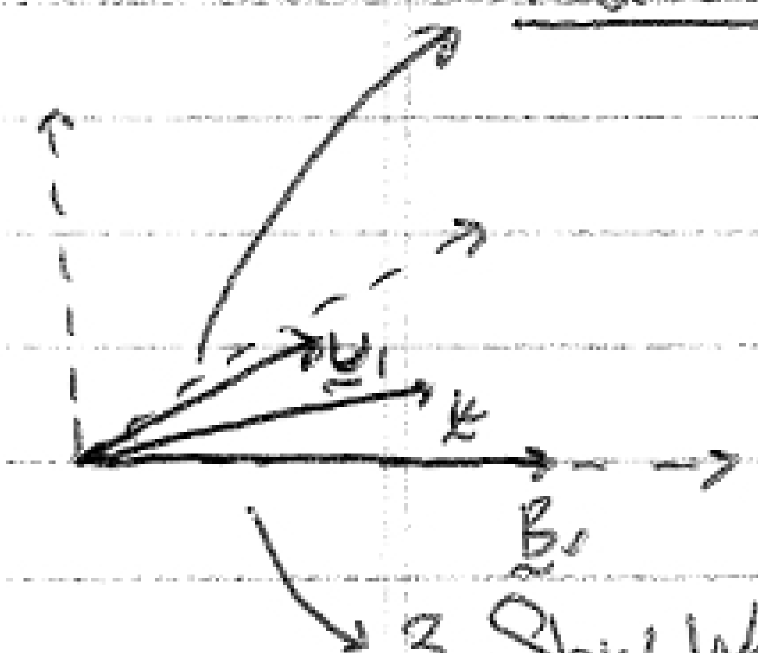


1. Alfven Waves: a. $\omega^2 = k_\parallel^2 V_A^2$

b. Motion out of the plane defined by $\underset{\sim}{B_0}$, $\underset{\sim}{k}$

c. Incompressible

d. Restoring Force: Magnetic Tension alone



2. Fast Waves: a. $\frac{\omega^2}{k^2} = \frac{1}{2}(C_S^2+V_A^2) + \frac{1}{2}\sqrt{(C_S^2+V_A^2)^2 - 4C_S^2 V_A^2\cos^2\theta}$

b. Motion in the plane of $\underset{\sim}{B_0}$ and $\underset{\sim}{k}$

c. Compressible (usually)

d. Restoring Force: i) Thermal and Magnetic Pressure Add!

ii) Magnetic Tension

3. Slow Waves: a. $\frac{\omega^2}{k^2} = \frac{1}{2}(C_S^2+V_A^2) - \frac{1}{2}\sqrt{(C_S^2+V_A^2)^2 - 4C_S^2 V_A^2\cos^2\theta}$

b. Motion in the plane of $\underset{\sim}{B_0}$ and $\underset{\sim}{k}$

c. Compressible

d. Restoring Force i) Thermal and Magnetic Pressure Subtract!

ii) Magnetic Tension

II. Polar Plot of MHD Wave Phase Speeds:

A. Dimensionless Version of MHD Dispersion Relation

1. Take $k_\parallel = k\cos\theta$ and $k_\perp = k\sin\theta$,
2. Normalize by dividing by ω_{ci}^6:

$$\left(\frac{\omega^2}{\omega_{ci}^2} - k_\parallel^2\frac{V_A^2}{\omega_{ci}^2}\right)\left[\frac{\omega^4}{\omega_{ci}^4} - \frac{\omega^2}{\omega_{ci}^2}(k_\perp^2 + k_\parallel^2)\frac{V_A^2}{\omega_{ci}^2}\left(1+\frac{C_s^2}{V_A^2}\right) + k_\parallel^2(k_\perp^2+k_\parallel^2)\frac{V_A^4}{\omega_{ci}^4}\frac{C_s^2}{V_A^2}\right] = 0$$

3. NOTE: a. Let $\tilde{\omega} = \frac{\omega}{\omega_{ci}}$

b. $$\frac{V_A^2}{\omega_{ci}^2} = \frac{\frac{B_0^2}{\mu_0\rho_0}}{\left(\frac{q_i^2 B_0^2}{m_i^2}\right)} = \frac{1}{\mu_0\epsilon_0}\left(\frac{\epsilon_0 m_i}{n_0 q_i^2}\right) = \frac{c^2}{\omega_{pi}^2}$$ ⇒ This is the ion inertial length.

DEFINE: $d_i \equiv \frac{c}{\omega_{pi}} = \frac{V_A}{\omega_{ci}}$

c. $$\frac{C_s^2}{V_A^2} = \left(\frac{\gamma p_0}{\rho_0}\right)\left(\frac{\mu_0\rho_0}{B_0^2}\right) = \frac{\gamma}{2}\frac{2\mu_0 p_0}{B_0^2} = \frac{\gamma}{2}\beta$$ ← Plasma β: $= \frac{\text{Thermal Press}}{\text{Magnetic Press.}}$

$$\boxed{\beta \equiv \frac{2\mu_0 p_0}{B_0^2}}$$

4. Thus,

$$\left(\tilde{\omega}^2 - k_\parallel^2 d_i^2\right)\left[\tilde{\omega}^4 - \tilde{\omega}^2(k_\perp^2 d_i^2 + k_\parallel^2 d_i^2)\left(1+\frac{\gamma}{2}\beta\right) + k_\parallel^2 d_i^2(k_\perp^2 d_i^2 + k_\parallel^2 d_i^2)\frac{\gamma}{2}\beta\right] = 0$$

5. There are only three parameters (dimensionless) which $\tilde{\omega}$ depends on:

$$\boxed{\tilde{\omega} = \tilde{\omega}_{MHD}(k_\perp d_i, k_\parallel d_i, \beta)}$$

a. Two define the parallel & perpendicular components of the wavevector (this is characteristic of most dispersion relations)

b. Only one other dimensionless parameter: $\boxed{\beta}$

6. NOTE: a. $d_i = \frac{\rho_i}{\sqrt{\beta_i}}$ where $\beta_i = \frac{2\mu_0 p_i}{B_0^2} = \frac{\beta}{2}$ for $T_i = T_e$ (true for MHD)

b. Thus, we could write $\tilde{\omega} = \tilde{\omega}_{MHD}(k_\perp\rho_i, k_\parallel\rho_i, \beta_i)$

II. A. (Continued)

7. Validity of MHD Approximation:

a. Remember $r_{Li} \ll L$, so if $L \sim \frac{1}{k}$, this means $\boxed{k r_{Li} \ll 1}$

b. Also

i. $v_0 = \frac{L}{T} \Rightarrow r_{Li} \ll L = T v_0$

ii. For $v_0 \sim v_{ti}$ and using $r_{Li} = \frac{v_{ti}}{\omega_{ci}}$, we get $\frac{\cancel{v_{ti}}}{\omega_{ci}} \ll T \cancel{v_{ti}}$

iii. Take $\omega \sim \frac{1}{T}$, giving us $\boxed{\omega \ll \omega_{ci}}$

c. Thus $\tilde{\omega} = \tilde{\omega}_{MHD}(k_\perp r_{Li}, k_\parallel r_{Li}, \beta_i)$ is valid when $\tilde{\omega} \ll 1$

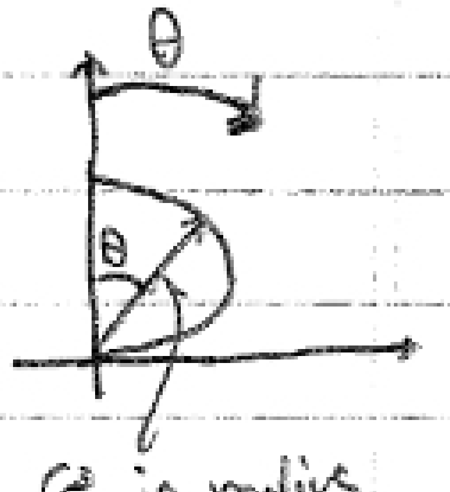
$k_\perp r_{Li}, k_\parallel r_{Li} \ll 1$.

B. Limits of $\frac{\omega}{k}$ at $\theta = 0$.

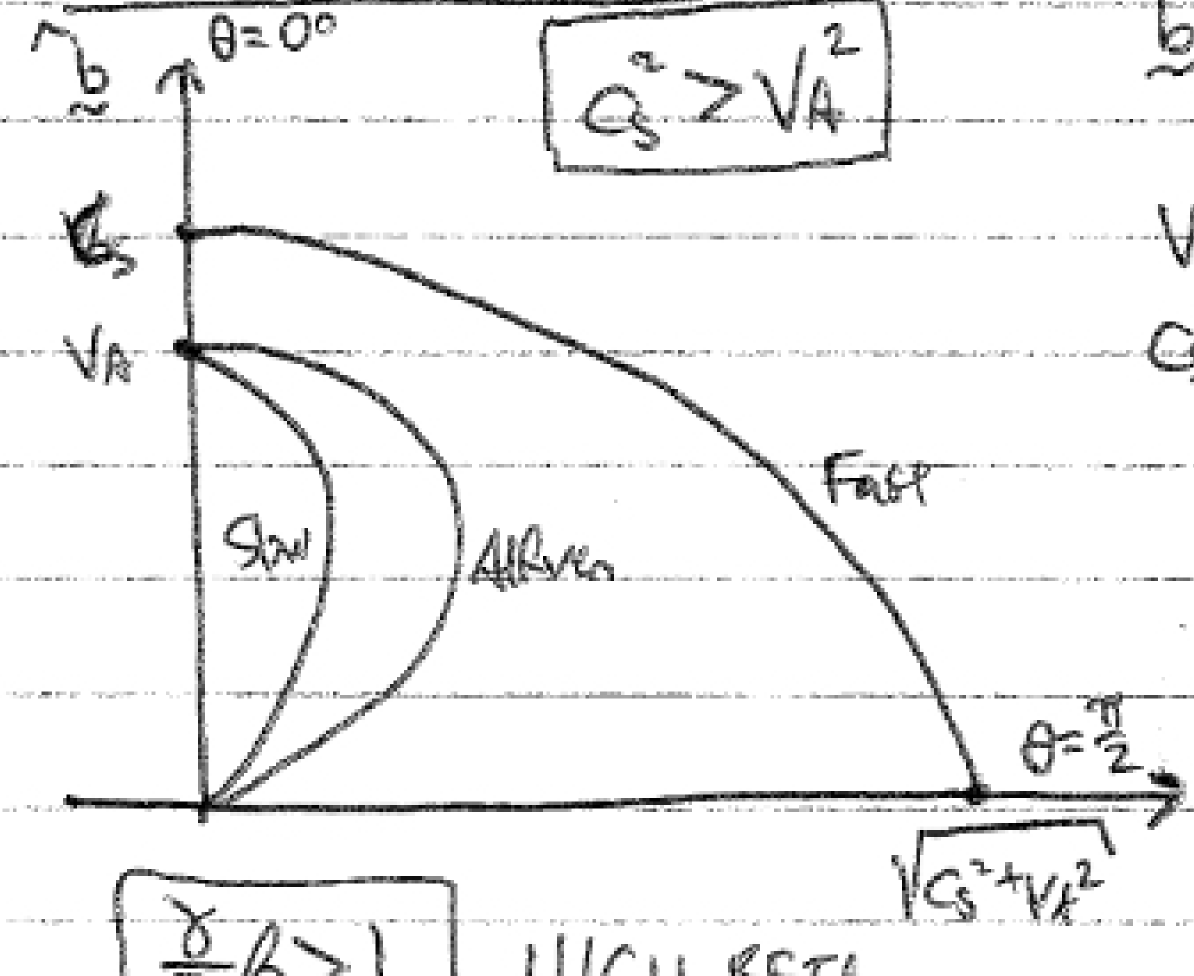
1. Phase velocity $v_p = \frac{\omega}{k}$ for waves at $\theta = 0$

	When $c_s^2 > v_A^2$:	When $c_s^2 < v_A^2$:
Fast	$\frac{\omega}{k} = c_s^2$	$\frac{\omega}{k} = v_A^2$
Alfven	$\frac{\omega}{k} = v_A^2$	$\frac{\omega}{k} = v_A^2$
Slow	$\frac{\omega}{k} = v_A^2$	$\frac{\omega}{k} = c_s^2$

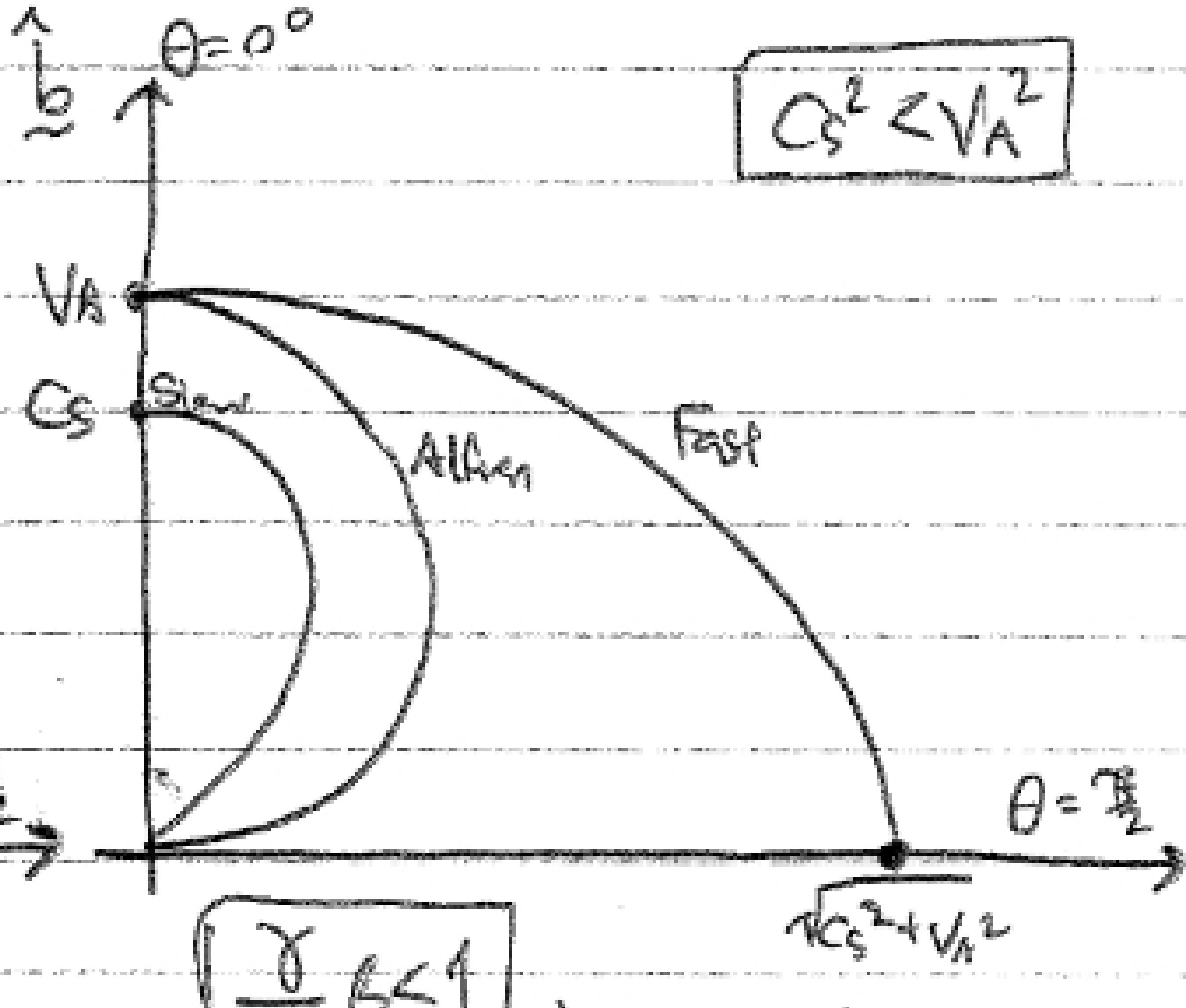
C. Polar Plots of $\frac{\omega}{k}$:



$\frac{\omega}{k}$ is radius



$\boxed{\frac{\gamma}{2}\beta > 1}$ HIGH BETA



$\boxed{\frac{\gamma}{2}\beta < 1}$ LOW BETA