I. *The center of mass of a system moves as if it were a single particle of mass equal to the total mass of the system, acted on by the total external force, and independent of the nature of the internal forces (as long as they follow $\mathbf{f}_{\alpha\beta} = -\mathbf{f}_{\beta\alpha}$, the weak form of Newton's Third Law).*

II. *The linear momentum of the system is the same as if a single particle of mass* M *were located at the position of the center of mass and moving in the manner the center of mass moves.*

III. *The total linear momentum for a system free of external forces is constant and equal to the linear momentum of the center of mass* (the law of conservation of linear momentum for a system).

IV. *The total angular momentum about an origin is the sum of the angular momentum of the center of mass about that origin and the angular momentum of the system about the position of the center of mass.*

V. *If the net resultant external torques about a given axis vanish, then the total angular momentum of the system about that axis remains constant in time.*

VI. *The total internal torque must vanish if the internal forces are central—that is, if $\mathbf{f}_{\alpha\beta} = -\mathbf{f}_{\beta\alpha}$, and the angular momentum of an isolated system cannot be altered without the application of external forces.*

VII. *The total kinetic energy of the system is equal to the sum of the kinetic energy of a particle of mass M moving with the velocity of the center of mass and the kinetic energy of motion of the individual particles relative to the center of mass.*

VIII. *The total energy for a conservative system is constant.*

Center of Mass Relations

$$M \equiv \sum_\alpha m_\alpha \qquad \vec{R} \equiv \frac{1}{M} \sum_\alpha m_\alpha \vec{r}_\alpha$$ [Weighted mean position]

Linear momentum and force:

$$\vec{P} \equiv \sum_\alpha \vec{p}_\alpha = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha = M \frac{d}{dt} \vec{R} = M \dot{\vec{R}}$$

$$\dot{\vec{P}} \equiv \sum_\alpha \dot{\vec{p}}_\alpha = \sum_\alpha m_\alpha \ddot{\vec{r}}_\alpha = M \ddot{\vec{R}}$$

$$\sum_\alpha \vec{F}_\alpha = \sum_\alpha \vec{F}^{(e)}_\alpha + \cancel{\sum_\alpha \sum_\beta \vec{f}_{\alpha\beta}} = \vec{F}^{(e)}_{tot}$$

[Assume $\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha}$ along their common line]

$$\text{So} \quad \dot{\vec{P}} = \sum_\alpha \dot{\vec{p}}_\alpha = \sum_\alpha \vec{F}_\alpha = \vec{F}^{(e)}_{tot} \Rightarrow \boxed{M\ddot{\vec{R}} = \vec{F}^{(e)}_{tot}}$$

Angular momentum and torque:

Note: $\sum_\alpha m_\alpha \vec{r}\,'_\alpha = 0, \quad \sum_\alpha m_\alpha \dot{\vec{r}}\,'_\alpha = 0$

$$\vec{L} \equiv \sum_\alpha \vec{L}_\alpha = \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha = \sum_\alpha \vec{r}_\alpha \times m \dot{\vec{r}}_\alpha$$

$$= \sum_\alpha (\vec{R} + \vec{r}\,'_\alpha) \times m_\alpha (\dot{\vec{R}} + \dot{\vec{r}}\,'_\alpha)$$

$$= \sum_\alpha m_\alpha \vec{R} \times \dot{\vec{R}} + \cancel{\sum_\alpha \vec{R} \times m_\alpha \dot{\vec{r}}\,'_\alpha} + \cancel{\sum_\alpha m_\alpha \vec{r}\,'_\alpha \times \dot{\vec{R}}} + \sum_\alpha \vec{r}\,'_\alpha \times m_\alpha \dot{\vec{r}}\,'_\alpha$$

$$= M \vec{R} \times \dot{\vec{R}} + \sum_\alpha \vec{r}\,'_\alpha \times \vec{p}\,'_\alpha$$

(About the CM)

$$\vec{L} = \vec{L}_{CM} + \vec{L}_{internal}$$

$$\dot{\vec{L}} \equiv \sum_\alpha \vec{r}_\alpha \times \dot{\vec{p}}_\alpha = \sum_\alpha \vec{r}_\alpha \times \vec{F}^{(e)}_\alpha + \cancel{\sum_\alpha \sum_\beta \vec{r}_\alpha \times \vec{f}_{\alpha\beta}}$$

[Note: $\vec{r}_\alpha \times \vec{f}_{\alpha\beta} + \vec{r}_\beta \times \vec{f}_{\beta\alpha} = (\vec{r}_\alpha - \vec{r}_\beta) \times \vec{f}_{\alpha\beta} = 0$]

$$\boxed{\dot{\vec{L}} = \vec{N}^{(e)}_{tot}} \qquad \vec{N}^{(e)}_{tot} \equiv \sum_\alpha \vec{N}^{(e)}_\alpha = \sum_\alpha \vec{r}_\alpha \times \vec{F}^{(e)}_\alpha$$

Kinetic energy

$$
\begin{aligned}
T &\equiv \sum_\alpha \tfrac{1}{2} m_\alpha \left(\dot{\vec{r}}_\alpha \cdot \dot{\vec{r}}_\alpha \right) \\
&= \sum_\alpha \tfrac{1}{2} m_\alpha \left(\dot{\vec{R}} + \dot{\vec{r}}_\alpha{}' \right) \cdot \left(\dot{\vec{R}} + \dot{\vec{r}}_\alpha{}' \right) \\
&= \sum_\alpha \tfrac{1}{2} m_\alpha \left(\dot{\vec{R}} \cdot \dot{\vec{R}} \right) + 2 \sum_\alpha \tfrac{1}{2} m_\alpha \dot{\vec{R}} \cdot \dot{\vec{r}}_\alpha{}' + \sum_\alpha \tfrac{1}{2} m_\alpha \left(\dot{\vec{r}}_\alpha{}' \cdot \dot{\vec{r}}_\alpha{}' \right) \\
&= \tfrac{1}{2} \left(\sum_\alpha m_\alpha \right) V^2 + \cancel{\left(\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}' \right)} \cdot \dot{\vec{R}} + \sum_\alpha \tfrac{1}{2} m_\alpha \left(\vec{v}_\alpha{}' \cdot \vec{v}_\alpha{}' \right) \\
&= \tfrac{1}{2} M V^2 + \sum_\alpha \tfrac{1}{2} m v_\alpha'^2
\end{aligned}
$$

$$\boxed{T = T_{CM} + T_{internal}}$$