Chapter 7: Lagrangian Mechanics

$L \equiv T - U$ (the Lagrangian)

Hamilton's principle $\delta \int L\,dt = 0$

$\downarrow$

Euler-Lagrange equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

$$\frac{\partial L}{\partial q_i} = 0 \iff \frac{\partial L}{\partial \dot{q}_i} = \text{constant}$$

$p_i \equiv \dfrac{\partial L}{\partial \dot{q}_i}$ "momentum conjugate to q_i"

Hamiltonian dynamics

$$H(q_i, p_i, t) = \sum_i p_i \dot{q}_i - L$$

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i} \end{cases}$$

First-order system of equations

Steps for solving problems in Lagrangian mechanics

① Write T and $\mathcal{U}$ in any convenient inertial frame.

② Choose suitable generalized coordinates q_i to describe the possible configurations

③ Write $L \equiv T - \mathcal{U}$ in terms of $q_i, \dot{q}_i, t$.

④ Work out the resulting Euler-Lagrange equations of motion.

(5) (Solve the equations, or answer the questions, as required.)

Hamiltonian Dynamics

Define $p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$

Solve (explicitly or implicitly) for

$\dot{q}_i = \dot{q}_i(q_i, p_i, t)$

Substitute into $L(q_i, \dot{q}_i, t)$ to get

$$H(q_i, p_i, t) \equiv \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \sum_i \dot{q}_i p_i - L$$

Note (LHS): $dH \equiv \sum\left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i\right) + \frac{\partial H}{\partial t} dt$

(RHS): $dH = \sum_i \left(\dot{q}_i dp_i + \cancel{p_i d\dot{q}_i} - \frac{\partial L}{\partial q_i} dq_i - \cancel{\frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i}\right) - \frac{\partial L}{\partial t} dt$

LHS $\Leftrightarrow$ RHS $\Rightarrow$

$$\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i}} \quad \text{and} \quad \boxed{\dot{p}_i = -\frac{\partial H}{\partial q_i}}$$

(using Euler-Lagrange: $\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$

$\Rightarrow \frac{d}{dt} p_i = \frac{\partial L}{\partial q_i}$)