Math 216 Differential Equations

Review of Complex Numbers

Introduction

This is a short review of the main concepts of *complex numbers*. Complex numbers are used throughout mathematics and its applications. In particular, when we try to solve differential equations it is often convenient and natural to use complex numbers to express the solutions. Here we review those ideas and results from the theory of complex numbers that will be used in Math 216.

A complex number z may be expressed as an ordered pair of *real* numbers:

$$z = (x, y) = x + iy$$

where $i := \sqrt{-1}$ (so $i^2 = -1$) and x and y are real numbers. The following notations are often used:

$x = \text{Re}(z)$ or $x = \Re(z)$ denotes the *real part* of the complex number z
$y = \text{Im}(z)$ or $y = \Im(z)$ denotes the *imaginary part* of the complex number z

Recall that two complex numbers are equal if and only if both the real and the imaginary parts are equal. In other words, $z_1 := (x_1, y_1)$ equals $z_2 := (x_2, y_2)$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

A convenient way of thinking about complex numbers is to imagine them as points in the (x, y) plane (in this case it is called the *complex plane*), as illustrated in the following figure. In the complex plane, the line $y = 0$ is frequently called the *real axis*, and the line

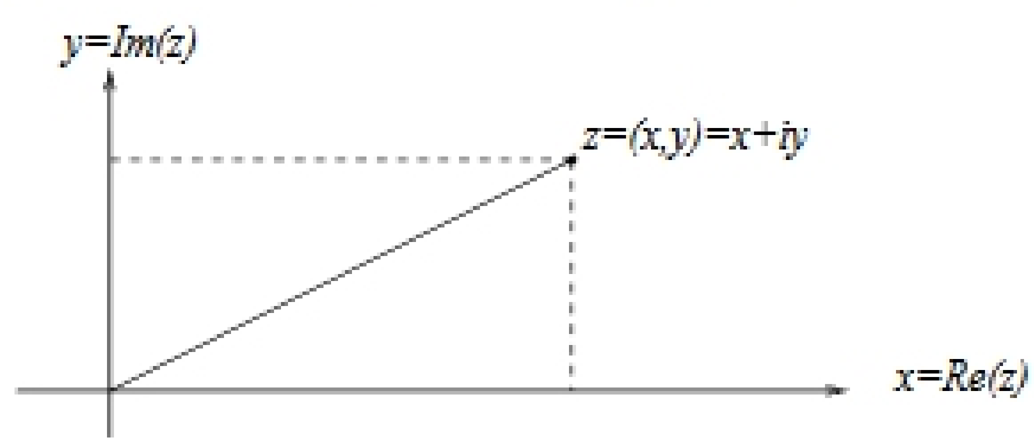

Figure 1: The complex number $z = x + iy$ plotted in the complex plane.

$x = 0$ is frequently called the *imaginary axis*.

Doing arithmetic with complex numbers

Addition and multiplication of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are defined by the following rules:

- Addition: $z_1 + z_2 := (x_1 + x_2, y_1 + y_2) = (x_1 + x_2) + i(y_1 + y_2)$.
- Multiplication: $z_1 z_2 := (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$.

Note that if we interpret z_1 and z_2 as points in the complex plane as in Figure 1, then addition of complex numbers is the same as vector addition in the plane; we are just adding the real and imaginary parts componentwise. On the other hand, the multiplication of two complex numbers may perhaps seem different than what you might have expected it to be; this is only an illusion, however, and when we introduce exponential forms for complex numbers later, the multiplication will make perfect sense.

Although complex numbers obey different rules of arithmetic than do ordinary real numbers, it is very important to keep in mind that the complex numbers simply generalize the notion of the real numbers. Indeed, we can think of the real number x as the complex number $(x, 0) = x + i0$. Such a complex number whose imaginary part is zero is said to be *purely real*. If we add or multiply two purely real complex numbers, then according to the rules for complex arithmetic, we have

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0) \quad \text{and} \quad (x_1, 0)(x_2, 0) = (x_1 x_2, 0)$$

so in each case the result is also a purely real complex number, and the real part in each case is exactly what we would have found by applying the usual rules of addition and multiplication for real numbers to the real parts. This shows that all the new operations defined for complex numbers when applied to purely real numbers give the usual familiar corresponding operations.

One way to think of $(0, 1)$ is as the *new* number i which is *purely imaginary* in the sense that its real part is zero, and so $(x, y) = x + iy$ is the sum of the purely real number x and the purely imaginary number iy.

Example: According to the above arithmetic rules for complex arithmetic, we have

$$(x, 0) + (0, y) = (x, y), \quad \text{and} \quad (0, 1)(y, 0) = (0, y).$$

Combining these, we deduce that

$$(x, y) = (x, 0) + (0, 1)(y, 0)$$

which is another way of writing the relation $z = x + iy$. □

Example: We can calculated repeated products of a complex number z with itself, which is what we mean by raising z to an integer power. Thus by definition really, $z^2 = zz$ and $z^3 = zzz$ and so on. In particular, $i^2 = ii = (0, 1)(0, 1)$. Using the rule for multiplication, we then see that

$$i^2 = ii = (0, 1)(0, 1) = (-1, 0) = -1$$

which verifies the fact that i is a square root of -1. □

Example: The fact that $i^2 = -1$ makes the rule for multiplication of complex numbers very easy to remember if one uses the $z = x + iy$ notation. Indeed just by multiplying out the individual terms,

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 + ix_1 y_2 + ix_2 y_1 + i^2 y_1 y_2$$

and then using $i^2 = -1$ we get

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

which is the rule for multiplication of complex numbers. □

It is easy to check directly from the definitions given of addition and multiplication of complex numbers that all of the familiar algebraic properties that we are familiar with hold for complex numbers too. In other words, complex arithmetic obeys the following rules:

- Commutative Law of Addition: $z_1 + z_2 = z_2 + z_1$
- Associative Law of Addition: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
- Commutative Law of Multiplication: $z_1 z_2 = z_2 z_1$
- Distributive Law: $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$
- Unique Additive Identity $0 = (0, 0) : z + 0 = 0 + z = z$
- Unique Multiplicative Identity $1 = (1, 0) : z \cdot 1 = 1 \cdot z = z$
- Additive inverse: $-z = (-x, -y) = -x - iy : z + (-z) = 0$
- Multiplicative inverse: For every complex number $z = (x, y) \neq 0$ there exists a complex number $w = (u, v)$ such that $(x, y)(u, v) = (u, v)(x, y) = (1, 0)$

It turns out that the multiplicative inverse of a nonzero complex number $z = (x, y)$ is the complex number

$$\left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2}\right)$$

which we denote by $1/z$. Now we can define the quotient of two complex numbers:

$$\frac{z_1}{z_2} := z_1 \cdot \frac{1}{z_2} = \frac{1}{z_2} \cdot z_1 \,.$$

Example: The multiplicative inverse of $i = (0, 1)$ is, according to the above formula,

$$\frac{1}{i} = (0, -1) = -i \,.$$

Therefore,

$$\frac{2}{i} = 2 \cdot \frac{1}{i} = -2i \,.$$

As a more complicated example, since

$$\frac{1}{1 + i} = \left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{2}i \,,$$