Hamiltonian Dynamics

Define $p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$

Solve (explicitly or implicitly) for

$$\dot{q}_i = \dot{q}_i(q_i, p_i, t)$$

Substitute into $L(q_i, \dot{q}_i, t)$ to get

$$H(q_i, p_i, t) \equiv \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \sum_i \dot{q}_i p_i - L$$

Note (LHS): $dH \equiv \sum \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right) + \frac{\partial H}{\partial t} dt$

(RHS): $dH = \sum_i \left(\dot{q}_i \, dp_i + \cancel{p_i \, d\dot{q}_i} - \frac{\partial L}{\partial q_i} dq_i - \cancel{\frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i} \right) - \frac{\partial L}{\partial t} dt$

LHS $\Leftrightarrow$ RHS $\Rightarrow$

$$\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i}} \quad \text{and} \quad \boxed{\dot{p}_i = - \frac{\partial H}{\partial q_i}}$$

(using Euler-Lagrange: $\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0$

$\Rightarrow \frac{d}{dt} p_i = \frac{\partial L}{\partial q_i}$)

Poisson Bracket "Commutator"

$$[f,g] \equiv \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right)$$

Note $[f,g] = -[g,f]$

Hamiltonian and total time derivative:

$$f = f(q_i, p_i, t)$$

$$\frac{d}{dt} f = \sum_i \left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i \right) + \frac{\partial f}{\partial t}$$

$$= \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \right) + \frac{\partial f}{\partial t}$$

$$= [f, H] + \frac{\partial f}{\partial t}$$

Note: $\frac{\partial q_i}{\partial q_k} = \delta_{ik}$, $\frac{\partial p_i}{\partial p_k} = \delta_{ik}$, $\frac{\partial q_i}{\partial p_k} = 0 = \frac{\partial p_i}{\partial q_k}$

$$\Rightarrow \quad [q_i, q_k] = 0$$

$$[p_i, p_k] = 0$$

$$[q_i, p_k] = \delta_{ik}$$

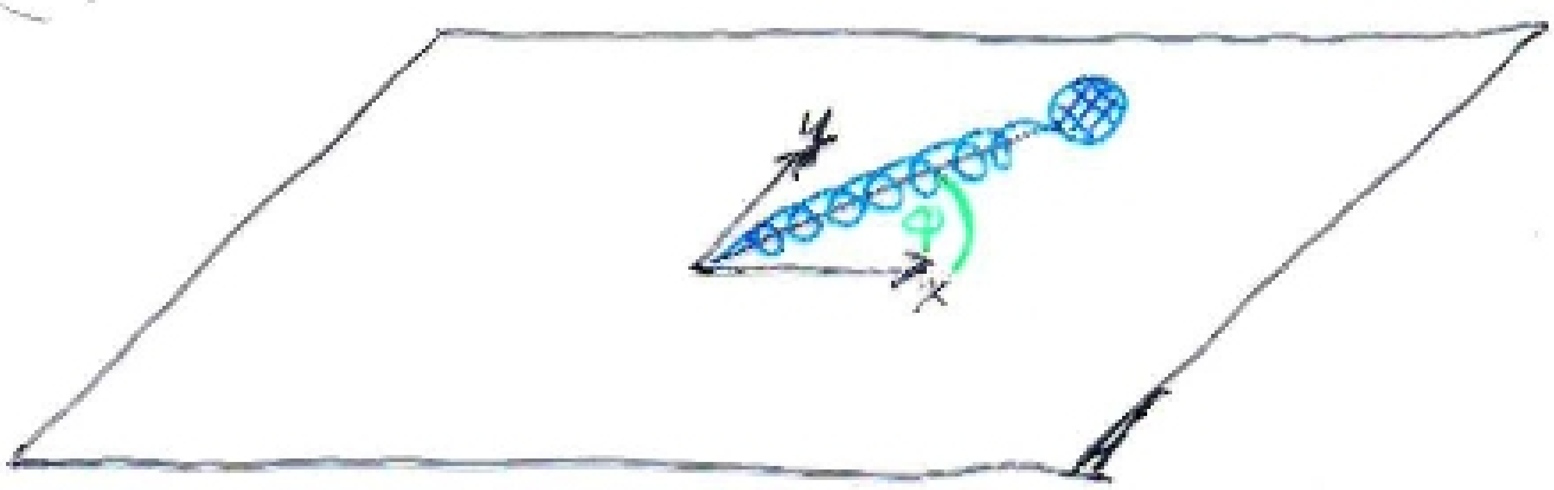
Functions f and g are "canonically conjugate" if $[f,g] = 1$.

p_i is canonically conjugate to q_i $\quad i = 1, 2, \ldots s$

Example problem: Pivoted spring with mass

Frictionless horizontal plane

Spring constant k, unstretched length b



Newton (Cartesian coordinates)

$$\vec{F} = -k(x - b\cos\varphi)\vec{i} - k(y - b\sin\varphi)\vec{j}$$

$$F_x = -kx\left(1 - \frac{b}{\sqrt{x^2+y^2}}\right)$$

$$F_y = -ky\left(1 - \frac{b}{\sqrt{x^2+y^2}}\right)$$

$$\begin{cases} m\ddot{x} = -kx\left(1 - \frac{b}{\sqrt{x^2+y^2}}\right) \\ m\ddot{y} = -ky\left(1 - \frac{b}{\sqrt{x^2+y^2}}\right) \end{cases}$$