ME 406
Orbital Stability

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sysid
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Mathematica  6.0.3, DynPac 11.01, 1/13/2009
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■ 1. Introduction

In this notebook we develop some examples which illustrate the concept of orbital stability. We begin by reviewing the "standard" definition of stability, which is usually called Liapunov stability. The system under consideration is an autonomous set of n first order differential equations:

$$\dot{X} = F(X) \quad ,$$

where X is the state vector and F is the slope function.

Here is the definition of Liapunov stability. Given a solution $\mathbf{X}^*(t)$ with a given initial value $\mathbf{X}^*(t_0)$, we say that $\mathbf{X}^*$is stable if, given any $\epsilon > 0$, we can find a $\delta > 0$ such that for any solution $\mathbf{X}(t)$ satisfying $|| \mathbf{X}^*(t_0) - \mathbf{X}(t_0)|| < \delta$, it is true that $|| \mathbf{X}^*(t) - \mathbf{X}(t)|| < \epsilon$ for all $t \geqslant t_0$. We call a stable solution $\mathbf{X}^*$ strictly stable if there exists an $\eta > 0$ such that $|| \mathbf{X}^*(t_0) - \mathbf{X}(t_0)|| < \eta$ implies that $|| \mathbf{X}^*(t) - \mathbf{X}(t)||$ goes to zero as $t \to \infty$.

This way of defining stability means that neighboring solutions evaluated at the same time must remain neighboring. We shall see below that for periodic solutions this is in general too restrictive a definition of stability. A more appropriate definition of stability for such systems will be given after our first example.

■ 2. Example 1 - A Nonlinear Center

We consider Duffing's equation with a hard spring, a parameter a, and no damping: $\ddot{x} + x + a\, x^3 = 0$.

We define the system for DynPac.

```
setstate[{x, y}]; setparm[{a}];
slopevec = {y, -x - a x^3}; sysname = "Duffing";
```

We set a parameter value of 0.5.

```
parmval = {0.5};
```

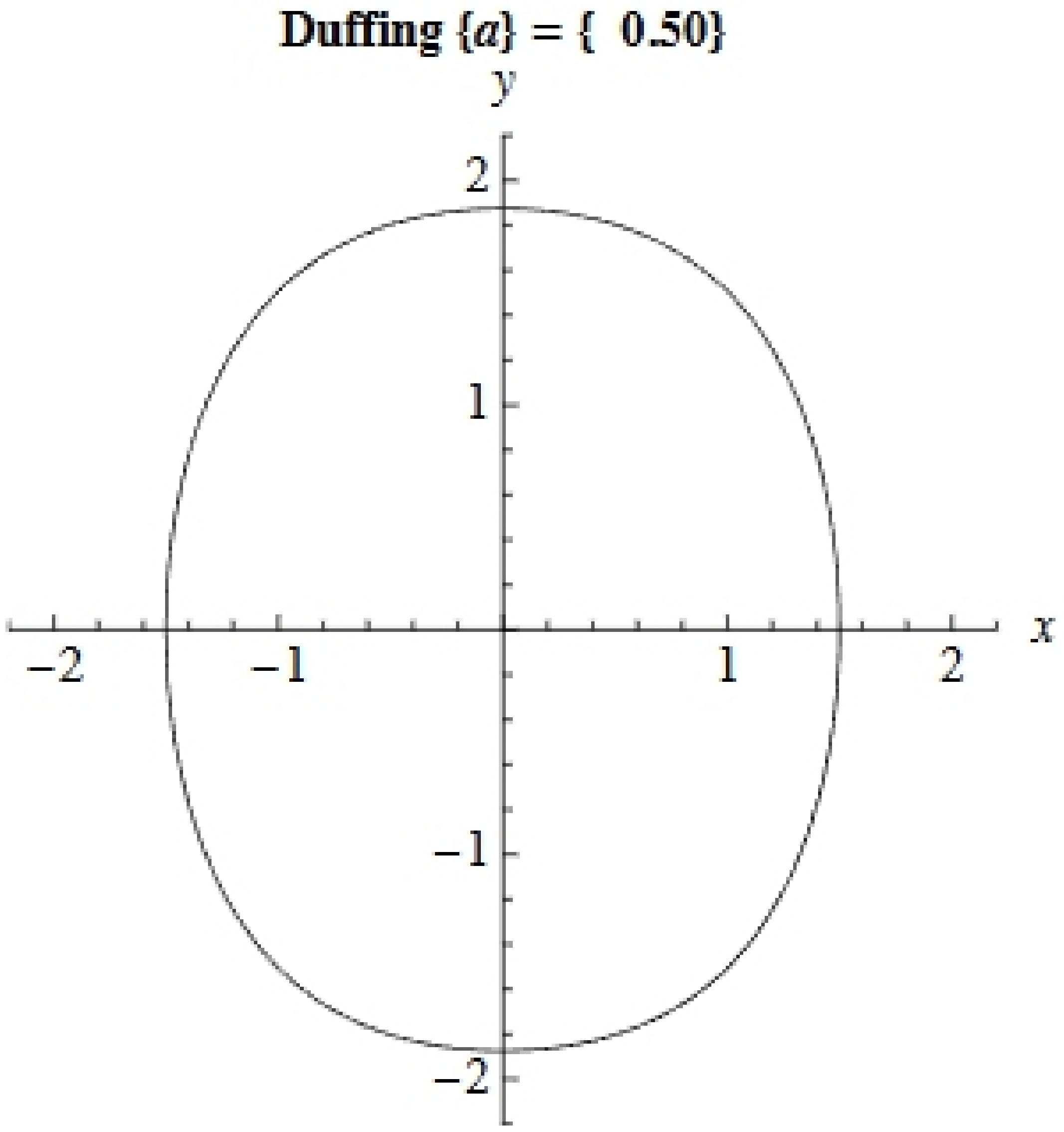
As we already know, all of the solutions of this equation for $a \geq 0$ are periodic. As an example, let's find and plot the solution with intial conditions (1.5, 0).

```
t0 = 0.0; h = 0.02; nsteps = 300; initvec = {1.5, 0};
```

```
sol1 = integrate[initvec, t0, h, nsteps];
```

```
asprat = 1.0; plrange = {{-2.2, 2.2}, {-2.2, 2.2}}; imsize = 250;
```

```
graph1 = phaser[sol1]
```



We find the period of the solution:

```
period[sol1]
```

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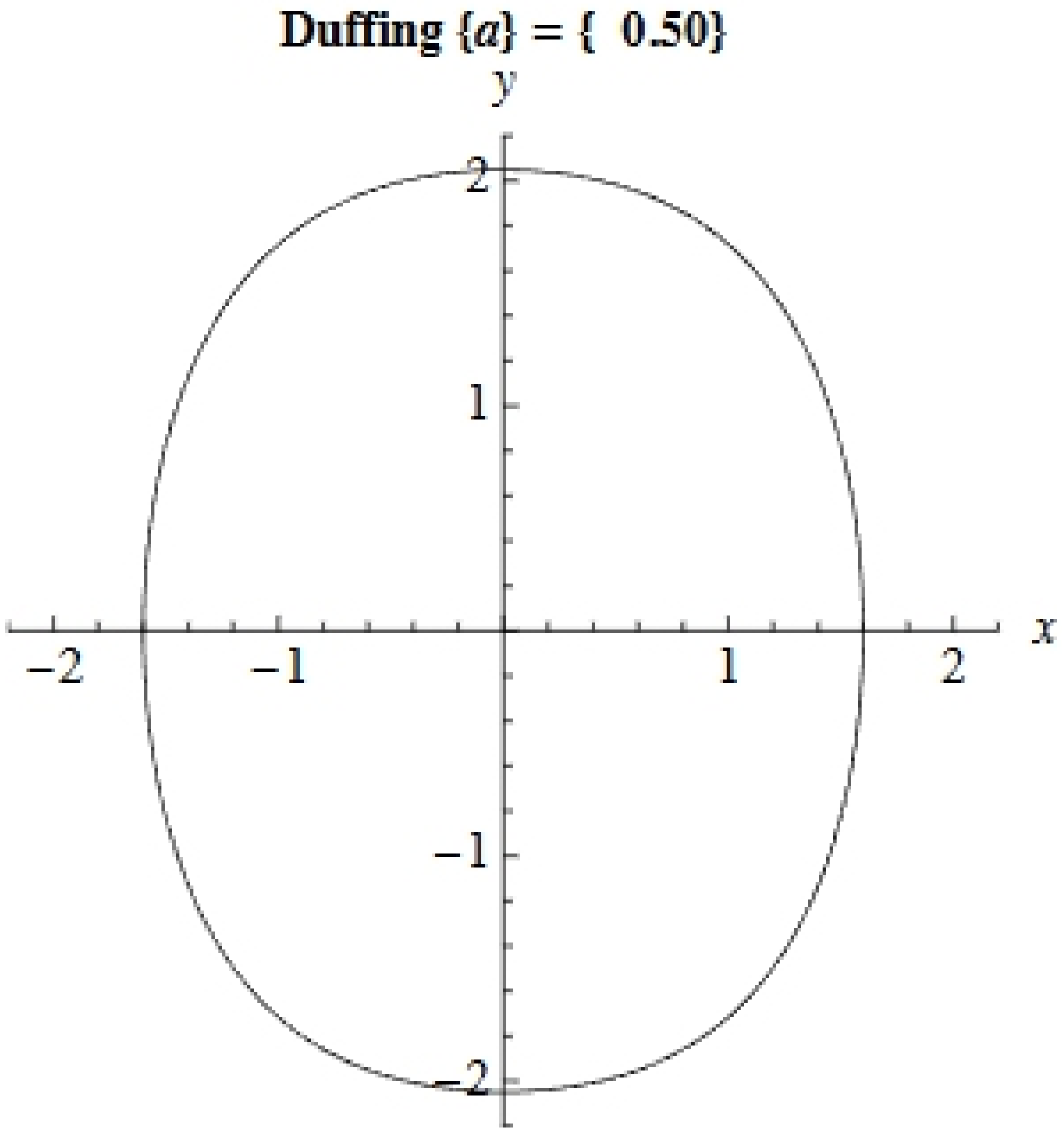
4.64
```

Now consider a nearby solution, obtained by altering the initial conditions slightly.

```
initvec = {1.6, 0}; sol2 = integrate[initvec, t0, h, nsteps];
```

```
graph2 = phaser[sol2]
```



We show the two graphs together.

```
graph3 = show[graph1, graph2]
```