MA 138 COURSE OUTLINE

DR. ANDREW SCHWARTZ, PH.D.

1.1 Propositional Logic (Propositional Calculus)

(1) whether or not something is indeed a proposition
(2) using basic propositions to construct new propositions using negation and logical connectives $\neg, \vee, \wedge, \oplus, \rightarrow$ (*or* $\Rightarrow$)
(3) constructing truth tables
(4) converse, contrapositive, and inverse
(5) bitwise operations and notation

1.2 Propositional Equivalences

(1) tautology - always true
(2) contradiction - always false
(3) contingency - neither of the above
(4) logical equivalence $\equiv$ (not a logical connective but rather a statement)
(5) common logical equivalences p. 24 & 25 (DeMorgan's Laws)

1.3 Predicates and Quantifiers

(1) predicate - think subject and predicates in English sentences
(2) propositional functions $P(x)$
(3) n-ary predicates $P(x_1, x_2, x_3, ...x_n)$
(4) preconditions and postconditions
(5) quantification (predicate calculus)
(6) quantifiers $\forall$ (universal), $\exists$ (existential), $\exists!$ (uniqueness)
(7) DeMorgan's Laws for Quantifiers

1.6 Introduction to Proofs

(1) theorem, propositions, facts, results
(2) proof
(3) axioms, postulates
(4) lemma, corollary, conjecture
(5) direct proof
(6) proof by contraposition (an indirect proof)
(7) vacuous proof (p is always false in $p \rightarrow q$)
(8) trivial proof (q is always true in $p \rightarrow q$)
(9) proof by contradiction (another indirect proof)
(10) counterexample
(11) mistakes in proof - begging the question, circular reasoning

1.7 Proof Methods and Strategy

(1) exhaustive proof and proof by cases
(2) without loss of generality
(3) existence proofs
(4) uniqueness proofs

4.1 Mathematical Induction

(1) mathematical induction - if we can can reach step 1, and we can reach the next step from a previous step, we can climb as high as we want
(2) basis step - show $P(1)$ is true
(3) inductive step - show that if $P(k)$ is true for all positive integers k, then $P(k+1)$ is true.

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(4) inductive hypothesis - $P(k)$

4.2 Strong Induction and Well-Ordering Principle

(1) strong induction - uses all the previous cases (not just the exact predecessor case) to prove the inductive step
(2) basis step - show $P(1)$ is true just as before
(3) inductive step - show that if $P(1), P(2), ..., P(k)$ are true for all positive integers k, then $P(k+1)$ is true.

2.1 Sets

(1) set - an unordered collection of objects
(2) paradoxes - logical inconsistencies
(3) set builder notation - $S = \{x \in \mathbb{Z}^- | x \text{ is even and } x^2 > 100\}$
(4) such that (s.t.) $\ni$
(5) blackboard bold (doublestruck)
    - $\mathbb{N} = \{0, 1, 2, 3, ...\}$ - the natural numbers
    - $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ - the integers (named for Zahlen, the German word for numbers, some attribute it also to Zermelo, a famous set theorist)
    - $\mathbb{Z}^+ = \{1, 2, 3, ...\}$ - the set of positive integers
    - $\mathbb{Q} = \{p/q | p, q \in \mathbb{Z}, \text{ and } q \neq 0\}$ - the set of rational numbers (the q is for quotient)
    - $\mathbb{R} = (-\infty, \infty)$ - the set of real numbers
    - $\mathbb{C} = \{a + bi | a, b \in \mathbb{R}, \text{ and } i^2 = -1\}$ - the set of complex numbers
    - $\mathbb{H} = \{a + bi + cj + dk | a, b, c, d \in \mathbb{R}, \text{ and } i^2 = j^2 = k^2 = ijk = -1\}$ - the set of quarternions (named for Sir William Rowan Hamilton)
    - $\mathbb{O}$ - the set of octonions
    - and the rabbit hole goes deeper
(6) universal set - all objects under consideration
(7) empty set (null set) - $\emptyset$ or $\{\}$
(8) subset - $\subseteq$
(9) $\forall$ sets $S, \emptyset \subseteq S$ and $S \subseteq S$
(10) Venn diagram
(11) proper subset - $\subset$
(12) cardinality - the size of a set (can be infinite or finite); denoted $|S|$ for a set S
(13) power set - $P(S)$
(14) Cartesian products - $A \times B$ (a relation more in Chapter 8)
(15) using set notation with quantifiers

2.2 Set Operations

(1) union $A \cup B$ - the set that contains those elements that are either in A or B, or in both
(2) intersection $A \cap B$ - the set that contains those elements that are in both A and B
(3) disjoint - sets whose intersection is the empty set
(4) difference $A - B$ or A/B - the containing those elements that are in A but not in B (also called the complement of B with respect to A)
(5) complement A^c, $\overline{A}$, or $U - A$ - the complement of with repect to the universal set U
(6) symmetric difference $A \bigoplus B$ or $A \Delta B$ - the set that contains those elements that are either in A or B, but not in both
(7) set identities p. 124
(8) generalized unions and intersections

2.3 Functions

(1) functions (mappings, transformations) $f : A \to B$ or $f : A \mapsto B$ - assignment of exactly one element of B to each element of A.
(2) domain, codomain (image, preimage)
(3) equality of functions (think of as a cartesian product)
(4) special functions (floor $\lfloor x \rfloor$, ceiling $\lceil x \lceil$, factorial $n!$)
(5) combinations of functions (addition, subtraction, multiplication, and division [when it is defined])

(6) one-to-one functions $1-1$ (injective) - $f(a)=f(b) \Rightarrow a=b$ for all a and b in the domain of f
(7) onto functions $\hookrightarrow$ (surjective) - $\forall b \in B, \exists a \in A \ni f(a)=b$
(8) increasing [strictly increasing] - if $f(x) \leq f(y)[f(x) < f(y)]$ whenever $x < y$ and x, y are in the domain of f
(9) decreasing [strictly decreasing] - if $f(x) \geq f(y)[f(x) > f(y)]$ whenever $x < y$ and x, y are in the domain of f
(10) one-to-one correspondence (bijective)
(11) inverse functions (invertible)
(12) compositions of functions
(13) graphs of functions - set of ordered pairs $\{(a, b) | a \in A \text{ and } f(a)=b\}$

2.4 Sequences and Summations

(1) sequence - function from a subset of the integers to a set S
(2) term - denoted a_n, the nth member of the sequence
(3) geometric progression - sequence of the form $a, ar, ar^2, ..., ar^n, ...$ with initial term a and common ratio r
(4) arithmetic progression - sequence of the form $a, a+d, a+2d, ..., a+nd, ...$ with initial term a and common difference d
(5) strings - another term for sequence, however usually written in form $a_1 a_2 ... a_n$
(6) length - number of terms in a string
(7) empty string - denoted λ or $\emptyset$, the string that has no terms
(8) summation notation - $\sum_{j=m}^{n} a_j$ or $\displaystyle\sum_{j=m}^{n} a_j$ or $\sum_{m \leq j \leq n} a_j$ to represent $a_m + a_{m+1} + ... + a_n$ (the last one is a misprint in the text)
(9) index of summation - j in the example above
(10) lower limit - n in the example above
(11) upper limit - m in the example above
(12) geometric series - sums of terms of a geometric progression
(13) $\displaystyle\sum_{j=0}^{n} ar^j = \frac{a(1-r^{n+1})}{1-r}$ when $r \neq 1$
(14) $\displaystyle\sum_{j=0}^{n} ar^j = \frac{a(1-r^{n+1})}{1-r}$ when $r = 1$
(15) common summation formulae - p. 157 Table 2
(16) cardinality - two sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B
(17) countable - sets that are either finite or has the same cardinality as the set of positive integers
(18) uncountable - reserved for when an infinite set is uncountable, denoted by $\aleph_0$ (pronounced aleph null or aleph naught [aleph is the first letter of the Hebrew alphabet])
(19) degrees of uncountablity - $\aleph_0, \aleph_1, \aleph_2, ...$ (subject of great mathematics debate)
(20) Cantor diagonalization argument - most common and widely accepted proof that the real numbers are not countable

8.1 Relations and Their Properties

(1) relation - a (binary) relation from a set A to a set B is a subset of $A \times B$
(2) related - when $(a, b) \in R$, we say that a is related to b, we use the notation aRb.
(3) relation on the set A - a relation from A to A
(4) reflexive - property of a relation where $(a, a) \in R$ for every element $a \in A$
(5) symmetric - property of a relation where $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$
(6) antisymmetric - property of a relation where $(a, b) \in R$ and $(b, a) \in R$, then $a=b$ for all $a, b \in A$
(7) transitive - property of a relation where whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for all $a, b, c \in A$
(8) combining relations - since relations can be thought of as merely sets, we can take unions, intersections, and differences just like any other set