GRAPHING

MATH 152, SECTION 55 (VIPUL NAIK)

Corresponding material in the book: Section 4.8

Difficulty level: Hard.

What students should definitely get: The main concerns in graphing a function, how to figure out what needs figuring out. It is important for students to go through all the graphing examples in the book and do more hands-on practice. Transformations of graphs. Quickly graphing constant, linear, quadratic graphs.

What students should hopefully get: How all the issues of symmetry, concavity, inflections, periodicity, and derivative signs fit together in the grand scheme of graphing. The qualitative characteristics of polynomial function and rational function graphs, as well as graphs involving a mix of trigonometric and polynomial functions.

Weird feature: Ironically, there are very few pictures in this document. The naive explanation is that I didn't have time to add many pictures. The more sophisticated explanation is that since the purpose here is to review how to graph functions, having actual pictures drawn perfectly is counterproductive. Please keep a paper and pencil handy and sketch pictures as you feel the need.

EXECUTIVE SUMMARY

0.1. Symmetry yet again.

Words...

(1) All mathematics is the study of symmetry (well, not all).
(2) One interesting kind of symmetry that we often see in the graph of a function is *mirror symmetry* about a vertical line. This means that the graph of the function equals its reflection about the vertical line. If the vertical line is $x = c$ and the function is f, this is equivalent to asserting that $f(x) = f(2c - x)$ for all x in the domain, or equivalently, $f(c + h) = f(c - h)$ whenever $c + h$ is in the domain. In particular, the domain itself must be symmetric about c.
(3) A special case of mirror symmetry is the case of an *even function*. An even function is a function with mirror symmetry about the y-axis. In other words, $f(x) = f(-x)$ for all x in the domain. (Even also implies that the domain should be symmetric about 0).
(4) Another interesting kind of symmetry that we often see in the graph of a function is *half-turn symmetry* about a point on the graph. This means that the graph equals the figure obtained by rotating it by an angle of π about that point. A point (c, d) is a point of half-turn symmetry if $f(x) + f(2c - x) = 2d$ for all x in the domain. In particular, the domain itself must be symmetric about c. If f is defined at c, then $d = f(c)$.
(5) A special case of half-turn symmetry is an odd function, which is a function having half-turn symmetry about the origin.
(6) Another symmetry is *translation symmetry*. A function is *periodic* if there exists $h > 0$ such that $f(x + h) = f(x)$ for all x in the domain of the function (in particular, the domain itself should be invariant under translation by h). If a smallest such h exists, then such an h is termed the period of f.
(7) A related notion is that of a function with *periodic derivative*. If f is differentiable for all real numbers, and f' is periodic with period h, then $f(x + h) - f(x)$ is constant. If this constant value is k, then the graph of f has a two-dimensional translational symmetry by (h, k) and its multiples.

Cute facts...

(1) Constant functions enjoy mirror symmetry about every vertical line and half-turn symmetry about every point on the graph (can't get better).

(2) Nonconstant linear functions enjoy half-turn symmetry about every point on their graph. They do not enjoy any mirror symmetry because they are everywhere increasing or everywhere decreasing.
(3) Quadratic (nonlinear) functions enjoy mirror symmetry about the line passing through the vertex (which is the unique absolute maximum/minimum, depending on the sign of the leading coefficient). They do not enjoy any half-turn symmetry.
(4) Cubic functions enjoy half-turn symmetry about the point of inflection, and no mirror symmetry. Either the first derivative does not change sign anywhere, or it becomes zero at exactly one point, or there is exactly one local maximum and one local minimum, symmetric about the point of inflection.
(5) Functions of higher degree do not necessarily have either half-turn symmetry or mirror symmetry.
(6) More generally, we can say the following for sure: a nonconstant polynomial of even degree greater than zero can have at most one line of mirror symmetry and no point of half-turn symmetry. A nonconstant polynomial of odd degree greater than one can have at most one point of half-turn symmetry and no line of mirror symmetry.
(7) If a function is continuously differentiable and the first derivative has only finitely many zeros in any bounded interval, then the intersection of its graph with any vertical line of mirror symmetry is a point of local maximum or local minimum. The converse does not hold, i.e., points where local extreme values are attained do *not* usually give axes of mirror symmetry.
(8) If a function is twice differentiable and the second derivative has only finitely many zeros in any bounded interval, then any point of half-turn symmetry is a point of inflection. The converse does not hold, i.e., points of inflection do *not* usually give rise to half-turn symmetries.
(9) The sine function is an example of a function where the points of inflection and the points of half-turn symmetry are the same: the multiples of π. Similarly, the points with vertical axis of symmetry are the same as the points of local extrema: odd multiples of $\pi/2$.
(10) For a periodic function, any translate by a multiple of the period of a point of half-turn symmetry is again a point of half-turn symmetry. (In fact, any translate by a multiple of half the period is also a point of half-turn symmetry).
(11) For a periodic function, any translate by a multiple of the period of an axis of mirror symmetry is also an axis of mirror symmetry. (In fact, translation by multiples of half the period also preserve mirror symmetry).
(12) A polynomial is an even function iff all its terms have even degree. Such a polynomial is termed an *even polynomial*. A polynomial is an odd function iff all its terms have odd degree. Such a polynomial is termed an *odd polynomial*.
(13) Also, the derivative of an even function (if it exists) is odd; the derivative of an odd function (if it exists) is even.

Actions ...

(1) Worried about periodicity? Don't be worried if you only see polynomials and rational functions. Trigonometric functions should make you alert. Try to fit in the nicest choices of period. Check if smaller periods can work (e.g., for $\sin^2$, the period is π). Even if the function in and of itself is not periodic, it might have a periodic derivative or a periodic second derivative. The sum of a linear function and a periodic function has periodic derivative, and the sum of a quadratic function and a periodic function has a periodic second derivative.
(2) Want to milk periodicity? Use the fact that for a periodic function, the behavior everywhere is just the behavior over one period translates over and over again. If the first derivative is periodic, the increase/decrease behavior is periodic. If the second derivative is periodic, the concave up/down behavior is periodic.
(3) Worried about even and odd, and half-turn symmetry and mirror symmetry? If you are dealing with a quadratic polynomial, or a function constructed largely from a quadratic polynomial, you are probably seeing some kind of mirror symmetry. For cubic polynomials and related constructions, think half-turn symmetry.
(4) Use also the cues about even and odd polynomials.

0.2. Graphing a function. Actions ...

(1) To graph a function, a useful first step is finding the domain of the function.

(2) It is useful to find the intercepts and plot a few additional points.
(3) Try to look for symmetry: even, odd, periodic, mirror symmetry, half-turn symmetry, and periodic derivative.
(4) Compute the derivative. Use that to find the critical points, the local extreme values, and the intervals where the function increases and decreases.
(5) Compute the second derivative. Use that to find the points of inflection and the intervals where the function is concave up and concave down.
(6) Look for vertical tangents and vertical cusps. Look for vertical asymptotes and horizontal asymptotes. For this, you may need to compute some limits.
(7) Connect the dots formed by the points of interest. Use the information on increase/decrease and concave up/down to join these points. To make your graph a little better, compute the first derivative (possibly one-sided) at each of these points and start off your graph appropriately at that point.

Subtler points... (see the "More on graphing" notes for an elaboration of these points; not all of them were covered in class):

(1) When graphing a function, there may be many steps where you need to do some calculations and solve equations and you are unable to carry them out effectively. You can skip some of the steps and come back to them later.
(2) If you cannot solve an equation exactly, try to approximate the locations of roots using the intermediate value theorem or other results such as Rolle's theorem.
(3) In some cases, it is helpful to graph multiple functions together, on the same graph. For instance, we may be interested in graphing a function and its second and higher derivatives. There are other examples, such as graphing a function and its translates, or a function and its multiplicative shifts.
(4) A graph can be used to suggest things about a function that are not obvious otherwise. However, the graph should not be used as conclusive evidence. Rather, the steps used in drawing the graph should be retraced and used to give an algebraic proof.
(5) We are sometimes interested in sketching curves that are not graphs of functions. This can be done by locally expressing the curve piecewise as the graph of a function. Or, we could use many techniques similar to those for graphing functions.
(6) For a function with a piecewise description, we plot each piece within its domain. At the points where the definition changes, determine the one-sided limits of the function and its first and second derivatives. Use this to make the appropriate open circles, asymptotes, etc.

1. GRAPHING IN GENERAL

The goal of this lecture is to make you more familiar with the tools and techniques that can be used to graph a function. The book has a list of points that you should keep in mind. The list in the book isn't complete – there are a number of additional points that tend to come up for functions of particular kinds, but it is a good starting point. But in this lecture, we'll focus on something more than just the techniques – we'll focus on the broad picture of why we want to draw graphs and what information about the function we want the graph to convey. Working from that, we will be able to reconstruct much of the book's strategy.

1.1. Graphs – utility, sketching and plotting. The graph of a function f on a subset of the real numbers is the set of points in $\mathbb{R}^2$ (the plane) of the form $(x, f(x))$, where x is in the domain of f. The graph of f gives a geometric description of f, and it completely determines f. For a given $x = x_0$, $f(x_0)$ is the y-coordinate of the unique point of the graph that is also on the line $x = x_0$.

Graphs are useful because they allow us to see many things about the function at the same time, and enable us to use our visual instincts to answer questions about the function. It is usually easy to look at the graph and spot, without precise measurement, phenomena such as periodicity, symmetry, increase, decrease, discontinuity, change in direction, etc. Thus, the graph of a function, *if correctly drawn*, is not only equivalent in information content to the function itself, it makes that information content much more easy to read.

The problem is with the caveat *if correctly drawn*. The domains of most of the functions we consider are unions of intervals, so they contain infinitely many points. *Plotting the graph* in a complete sense would involve evaluating the function at these infinitely many points. In practice, *graph plotting* works by dividing the domain into very small intervals (say, of length 10^{-3}), calculating the values of the function (up to some