Modern Physics (PHY 3305) Lecture Notes

Bound States (Ch. 5.3-5.5)

SteveSekula, 18 February 2010 (created 13 December 2009)

Review

tags: lecture

- We discussed the meaning of the SWE and considered the simplest solutions to it: free particles (free meaning no external forces)
- We then discussed the implication for knowledge, specifically the Uncertainty Principle
- We started considering "bound states" - problems of particles under the influence of constraints/forces

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Stationary States Revisited: Separation of Variables

Our first step is to separate the space and time parts of the wave into separate functions, multiplied times one another:

$$\Psi(x,t) = \psi(x)\phi(t)$$

This is an assumption, but making it allows us to try to simplify the problem and test our solutions. It does reduce the generality of our solutions, but its advantage is a practical one: these special solutions are often of great interest (and utility!).

We can now re-write the SWE:

$$-\frac{\hbar^2}{2m}\phi(t)\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x)\phi(t) = i\hbar\psi(x)\frac{d\phi(t)}{dt}$$

and then re-order the terms to achieve separation:

$$-\frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{d^2\psi(x)}{dx^2}+U(x)=i\hbar\frac{1}{\phi(t)}\frac{d\phi(t)}{dt}$$

What does separating the variables mean? It means that we assume that x is not affected by t and t is not affected by x. That means that if time changes, the left-hand side of the SWE above DOES NOT. If that side of the equation is constant, then the right side MUST be constant. Thus:

$$-\frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{d^2\psi(x)}{dx^2}+U(x)=i\hbar\frac{1}{\phi(t)}\frac{d\phi(t)}{dt}=C$$

C is the "separation constant". Can we constrain this constant?

Yes: Let us now consider each part of the equation separately.

Stationary States: the temporal part, $\phi(t)$

This is:

$$i\hbar\frac{1}{\phi(t)}\frac{d\phi(t)}{dt}=C$$

or

$$\frac{d\phi(t)}{dt}=-\frac{iC}{\hbar}\phi(t)$$

NOTA BENE: Eqn. 5-6 in Harris is MISSING the minus sign!

A solution to this equation is:

$$\phi(t)=e^{-i(C/\hbar)t}$$

What does all this mean? Write the solution in terms of the Euler Equation:

$$e^{-i(C/\hbar)t} = cos((C/\hbar)t) - isin((C/\hbar)t)$$

We see that $C/\hbar$ represents a pure frequency (e.g. $2\pi f$). That means $C = 2\pi\hbar f = \hbar\omega = E$.

This means that when we separate variables, we are in fact FOCUSING ON STATES WITH *WELL-DEFINED* ENERGIES. The separation constant IS that energy.

According to the separation of our original wave function, we can now write:

$$\Psi(x,t) = \psi(x)e^{-i(E/\hbar)t}$$

for the wave function. We haven't considered interactions with the potential yet, so $\psi(x)$ is still general and unsolved-for.

DISCUSSION: what is the probability density of this wave function?

Answer:

$$\Psi^*(x,t)\Psi(x,t) = \left(\psi^*(x)e^{i(E/\hbar)t}\right)\left(\psi(x)e^{-i(E/\hbar)t}\right) = \psi^*(x)\psi(x)$$

- Is there time dependence in the probability?
  - No - it disappears under the case we can separate space and time components of the wave function
  - The properties of such objects do not change in time - they are "stationary states"
- What are the implications for, say, electrons in an atom?
  - The electron is bound by the coulomb force to the atom. That potential is time independent. Classically, as it whizzes around the nucleaus is should be losing energy. But quantum mechanics says that's not the case: it tells us that the electron can appear in many places around the atom ($\psi(x)$), but its energy is constant and well-defined. The electron is not orbiting, in the classical sense, but rather in a probability cloud around the nucleus. If the probability density is constant, the charge density is constant, and if the charge density is constant, EM tells us it radiates no energy.