Lesson 10: The Vertex of a Parabola

(Cover 3.2 & Completing the Square)	**Read:** Section 4.1
Announce/Remind: Gateway, Gateway, Gateway, Gateway, Gateway, Gateway!	**Do:** WebWork, Team Homework, **Gateway**

Midterm 1 - Monday, October 10, 6-7:30 pm (Note: NOT Michigan time) Location TBA
Material to be covered: Sections 1.1-1.5, 2.1-2.5, 3.1-3.2, 4.1-4.3

The most important points and skills for §3.2

- Students are able to recognize formulas for quadratic functions in standard ($y = ax^2 + bx + c$), vertex ($y = a(x - h)^2 + k$) and factored ($y = a(x - r_1)(x - r_2)$) forms and use these forms to determine information about the graph of the function (y–intercept from standard form, vertex from vertex form, and x–intercept(s) from factored form).
- Given a graph of a quadratic function $f(x)$ that shows the coordinates of the vertex and one other point on the graph, students are able to find an explicit formula for $f(x)$ using vertex form.
- Given an explicit formula for a quadratic function, students are able to determine the x– and y–coordinates of the vertex as well as the formula for the axis of symmetry of the parabola by completing the square and then noting the values of h and k that are present in the vertex form $y = a(x - h)^2 + k$.
- Students should be able to determine the maximum/minimum of a quadratic function by converting the formula into vertex form and interpreting the meaning of the x– and y–coordinates of the vertex in terms of the "real-world" context described in word problems.

Comment:
Most students are already very familiar with quadratic functions, standard form, and factoring; however, completing the square is quite difficult for many students. When you do examples of completing the square, avoid the temptation to cut corners and *always* describe your calculation methods IN FULL and *put all steps in your boardwork.*

Suggested Lesson Plan:

0-20 Use this time for a quiz and/or to return graded work (quizzes or team homework) and/or to do more examples of the material from the previous lesson. Summarize the previous lesson in one or two sentences. Then outline the lesson for today.

20–30 Remind the students about the standard equation ($y = ax^2 + bx + c$) and factored form ($y = a(x - r_1)(x - r_2)$, where r_1 and r_2 are zeros) and what it means to know the zeros of a function. Introduce the third way of writing a quadratic function: vertex form ($y = a(x-h)^2+k$). Sketch the graph of $g(x) = 2(x - 3)^2 - 5$ and point out how to find the vertex of the parabola and the maximum value of the function from the formula.
Point out the key features of a graph that can be easily seen in each form (y–intercept in standard form, x–intercept(s) in factored form, and vertex in vertex form). Make sure to define the *vertex* and *axis of symmetry* as well as how these may be found in an equation in vertex form.

30–40 When you have established all of the terms, have the students practice finding a formula for a quadratic function when given the graph. Give the students two examples to try in their groups. In the first example, you could start with a graph that shows two x–intercepts and the coordinates of one other point (such as **Chapter 3 Review #19 on page 118**). For the second example, you could start with a graph that shows the coordinates of the vertex and the coordinates of one other point on the graph (such as **Chapter 3 Review #20 on page 118**). Circulate as the groups work. Some questions that you might find helpful to ask the students are:

- Based on the information that you can see on the graph, which way of writing a quadratic function would be easiest to use here?
- How do the points you can see on the graph help you to figure out the numbers that need to go into the formula?
- What added information do we gain from the extra point (i.e. not the zeros or vertex) on the graph?
- How can you know the sign of the leading coefficient just by looking at the graph?

Generally, students understand how to use zeros and/or vertex points in equations, but often have a difficult time using the "extra" point to solve for the leading coefficient a.

40–55 Move into a discussion of *how to find the vertex* of a parabola by "completing the square". Note: The discussion of completing the square in the Chapter 3 Skills section (pages 125–127) of the textbook gives a formula, but we don't want to emphasize this. Rather, we want the students to learn an algorithmic approach such as the one used in Examples 1 and 2 (pages 125-127); also Examples 2 an 4 from pages 112 and 114.

As an example, use $h(x) = 2x^2 - 12x + 13$ (as this is actually $g(x)$ in standard form).

If you really want to break down the technique into bite-sized pieces, here is one procedure that you could follow (for $h(x) = 2x^2 - 12x + 13$). This is the procedure used in the book.

Step 1: Factor the leading coefficient out of every term in the function.

$$h(x) = 2[x^2 - 6x + 13/2]$$

Step 2: Look at the number preceding the x–term. Divide this number by 2 and then square that value. number = -6 half the number = -3 which when squared is $(-3)^2 = 9$.

Step 3: Add and subtract the value you computed in Step 2 in between the x–term and constant term.

$$h(x) = 2[x^2 - 6x + \mathbf{9} - \mathbf{9} + 13/2]$$

Step 4: Group together the first three terms to have a perfect square.

$$h(x) = 2[(x^2 - 6x + 9) - 9 + 13/2] = 2[(x - 3)^2 - 9 + 13/2]$$

Step 5: Combine the constant terms left over outside the perfect square.

$$h(x) = 2[(x - 3)^2 - 5/2]$$

Step 6: Distribute the coefficient you factored out in Step 1.

$$h(x) = 2(x - 3)^2 - 5.$$

The most common mistakes are (i) not factoring the leading coefficient a out of everything in Step 1 and (ii) not distributing a correctly in Step 6.

Another approach (which lets the original constant term "come along for the ride"):

Step 1: Factor the leading coefficient out of every *non-constant* term in the function.

$$h(x) = 2[x^2 - 6x] + 13$$

Step 2: Look at the number preceding the x–term. Divide this number by 2 and then square that value. number = -6 half the number = -3 which when squared is $(-3)^2 = 9$.

Step 3: Add and subtract the value you computed in Step 2 in between the x–term and constant term.

$$h(x) = 2[x^2 - 6x + \mathbf{9} - \mathbf{9}] + 13$$

Step 4: Group together the first three terms to have a perfect square.

$$h(x) = 2[(x^2 - 6x + 9) - 9] + 13 = 2[(x-3)^2 - 9] + 13$$

Step 5: Distribute the coefficient you factored out in Step 1.

$$h(x) = 2(x-3)^2 - 18 + 13.$$

Step 6: Combine the constant terms left over outside the perfect square.

$$h(x) = 2(x-3)^2 - 5$$

There are certainly other algorithms. One of these is the following:

<u>Alternate method for completing the square</u>
Shylynn Lofton, Fall 2009

Let $f(x) = -10x^2 + 20x + 97$. Since the leading coefficient is not 1, factor it out of the first two terms, only. We do this because we are interested in forcing the group of variable expressions into a perfect square. The constant term is not altered in any way. I instruct my students to leave some space inside and outside of the parentheses. The space inside is used to 'complete the square.' The outside space is used to ensure that the expression is balanced.

$f(x) = -10x^2 + 20x + 97$

$= -10(x^2 - 2x \quad) \quad + 97$

$= -10(x^2 - 2x + 1) + 10 + 97$

$= -10(x^2 - 2x + 1) + 10 + 97$

$= -10(x-1)^2 + 107$

We complete the square by adding 1 to the 'grouping.' But in order to ensure that the value of the expression remains the same, we have to remember that we initially factored out −10. So, although we added 1 to the 'grouping,' we really added −10 to the expression. To balance that addition of −10, we must add +10. This is the point where errors could occur. I constantly remind my students to do a quick mental check of the expression's value. I also remind them to think about the steps that led up to this point. With practice, most students have no problems with this method.

Pick the method that you think students will be most comfortable with. You should pick ONE method and STICK WITH IT. (You should certainly encourage students to use a process that makes sense to them, but YOU should always use the same algorithm so that students have a model to follow.)

55–65 Have the students try a few examples of completing the square. Pick a few exercises from **Chapter 3 Skills Refresher "Completing the Square" Tools Problems #13-22 on page 128** for the students to try. A problem like #22 can be a good one to do since the students must factor out a negative number.