4.5 Undetermined Coefficients II

The study of undetermined coefficients continues. Recorded here are special methods for efficiently solving the easily-solved equations. **Linear algebra is not required** in any of the special methods: only calculus and college algebra are assumed as background.

The special methods provide a justification for the basic trial solution method of the preceding section.

Library of Special Methods

It is assumed that the differential equation is already in easily-solved form: the library methods are designed to apply directly. If an equation requires a decomposition into easily-solved equations, then the desired solution is then the sum of the answers to the decomposed equations.

Equilibrium and Quadrature Methods. The special case of $ay'' + by' + cy = k$ where k is a constant occurs so often that an efficient method has been isolated to find y_p. It is called the **equilibrium method**, because in the simplest case y_p is a constant solution or an *equilibrium solution*. The method in words:

> Verify that the right side of the differential equation is constant. Cancel on the left side all derivative terms except for the lowest order and then solve for y by quadrature.

The method works to find a solution, because if a derivative $y^{(n)}$ is constant, then all higher derivatives $y^{(n+1)}$, y^{n+2}, etc., are zero. A precise description follows for second order equations.

Differential Equation	Cancelled DE	Particular Solution
$ay'' + by' + cy = k$, $c \neq 0$	$cy = k$	$y_p = \dfrac{k}{c}$
$ay'' + by' = k$, $b \neq 0$	$by' = k$	$y_p = \dfrac{k}{b}x$
$ay'' = k$, $a \neq 0$	$ay'' = k$	$y_p = \dfrac{k}{a}\dfrac{x^2}{2}$

The equilibrium method also applies to nth order linear differential equations $\sum_{i=0}^{n} a_i y^{(i)} = k$ with constant coefficients $a_0, \ldots, a_n$ and constant right side k.

A special case of the equilibrium method is the *simple quadrature method*, illustrated in Example **??**, page **??**. This method is used in elementary physics courses to solve falling body problems.

The Polynomial Method. The method applies to find a particular solution of $ay'' + by' + cy = p(x)$, where $p(x)$ represents a polynomial of degree $n \geq 1$. Such equations **always have a polynomial solution**; see Theorem **??**, page **??**.

Let a, b and c be given with $a \neq 0$. Differentiate the differential equation successively until the right side is constant:

$$
(1) \qquad
\begin{array}{lllllll}
ay'' & + & by' & + & cy & = & p(x), \\
ay''' & + & by'' & + & cy' & = & p'(x), \\
ay^{iv} & + & by''' & + & cy'' & = & p''(x), \\
 & & & & & \vdots & \\
ay^{(n+2)} & + & by^{(n+1)} & + & cy^{(n)} & = & p^{(n)}(x).
\end{array}
$$

Apply the equilibrium method to the *last equation* in order to find a polynomial **trial solution**

$$y(x) = d_m \frac{x^m}{m!} + \cdots + d_0.$$

It will emerge that $y(x)$ always has $n+1$ terms, but its degree can be either n, $n+1$ or $n+2$. The **undetermined coefficients** $d_0, \ldots, d_m$ are resolved by setting $x = 0$ in equations (**??**). The Taylor polynomial relations $d_0 = y(0), \ldots, d_m = y^{(m)}(0)$ give the equations

$$
(2) \qquad
\begin{array}{lllllll}
ad_2 & + & bd_1 & + & cd_0 & = & p(0), \\
ad_3 & + & bd_2 & + & cd_1 & = & p'(0), \\
ad_4 & + & bd_3 & + & cd_2 & = & p''(0), \\
 & & & & & \vdots & \\
ad_{n+2} & + & bd_{n+1} & + & cd_n & = & p^{(n)}(0).
\end{array}
$$

These equations can always be solved by **back-substitution**; linear algebra is not required. Three cases arise, according to the number of zero roots of the characteristic equation $ar^2 + br + c = 0$. The values $m = n, n+1, n+2$ correspond to zero, one or two roots $r = 0$.

No root $r = 0$. Then $c \neq 0$. There were n integrations to find the trial solution, so $d_{n+2} = d_{n+1} = 0$. The unknowns are d_0 to d_n. The system can be solved by simple back-substitution to uniquely determine $d_0, \ldots, d_n$. The resulting polynomial $y(x)$ is the desired solution $y_p(x)$.

One root $r = 0$. Then $c = 0$, $b \neq 0$. The unknowns are $d_0, \ldots, d_{n+1}$. There is no condition on d_0; simplify the trial solution by taking $d_0 = 0$. Solve (**??**) for unknowns d_1 to d_{n+1}, as in the no root case.

Double root $r = 0$. Then $c = b = 0$ and $a \neq 0$. The equilibrium method gives a polynomial trial solution $y(x)$ involving $d_0, \ldots, d_{n+2}$. There are no conditions on d_0 and d_1. Simplify y by taking $d_0 = d_1 = 0$. Solve (**??**) for unknowns d_2 to d_{n+2}, as in the no root case.

College algebra back-substitution applied to (**??**) is illustrated in Example **??**, page **??**. A complete justification of the polynomial method appears in the proof of Theorem **??**, page **??**.

Recursive Polynomial Hybrid.

A *recursive method* based upon quadrature appears in Example **??**, page **??**. This method, independent from the *polynomial method* above, is useful when the number of equations in (**??**) is two or three.

Some researchers (see [Gupta]) advertise the recursive method as easy to remember, easy to use and faster than other methods. In this textbook, the method is advertised as a **hybrid**: equations in (**??**) are written down, but equations (**??**) are not. Instead, the undetermined coefficients are found recursively, by repeated quadrature and back-substitution.

Classroom testing of the recursive polynomial method reveals it is best suited to algebraic helmsmen with flawless talents. The method should be applied when conditions suggest rapid and reliable computation details. Error propagation possibilities dictate that polynomial solutions of degree 4 or larger be subjected to an answer check.

Polynomial × Exponential Method.

The method applies to special equations $ay'' + by' + cy = p(x)e^{kx}$ where $p(x)$ is a polynomial. The idea, due to Kümmer, uses the transformation $y = e^{kx}Y$ to obtain the auxiliary equation

$$[a(D+k)^2 + b(D+k) + c]Y = p(x), \quad D = \frac{d}{dx}.$$

The polynomial method applies to find Y. Multiplication by e^{kx} gives y.

Computational details are in Example **??**, page **??**. Justification appears in Theorem **??**. In words, to find the differential equation for Y:

> In the differential equation, replace D by $D+k$ and cancel e^{kx} on the RHS.

Polynomial × Exponential × Cosine Method.

The method applies to equations $ay'' + by' + cy = p(x)e^{kx}\cos(mx)$ where $p(x)$ is a polynomial. Kümmer's transformation $y = e^{kx}\,\mathcal{R}\mathrm{e}(e^{imx}Y)$ gives the auxiliary problem

$$[a(D+z)^2 + b(D+z) + c]Y = p(x), \quad z = k + im, \quad D = \frac{d}{dx}.$$

The polynomial method applies to find Y. Symbol $\mathcal{R}\mathrm{e}$ extracts the real part of a complex number. Details are in Example **??**, page **??**. The formula is justified in Theorem **??**. In words, to find the equation for Y:

> In the differential equation, replace D by $D+k+im$ and cancel $e^{kx}\cos mx$ on the RHS.