MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.003: Signals and Systems — Spring 2004

TUTORIAL 9

Monday, April 12 and Tuesday, April 13, 2004

Announcements

- There is no problem set due this week.
- Quiz 2 will be held on **Thursday, April 15**, 7:30–9:30 p.m. in Walker Memorial. The quiz will cover material in Chapters 1–7 of O&W through Section 7.4, Lectures and Recitations through April 2, Problem Sets #1–6, and that part of Problem Set #7 involving problems from Chapter 7.
- The TAs will jointly hold office hours from 2–8 p.m. on Wednesday, April 14 and again from 10 a.m.–3 p.m. on Thursday, April 15. A schedule is posted on the 6.003 web site.
- A quiz review package is available on the 6.003 web site. TAs will hold two identical optional quiz review sessions on Monday, April 12 and Tuesday, April 13, 7:30–9:30 p.m. in 34-101.
- Because of the Patriot's Day holiday next week, there will be no tutorials next Monday and Tuesday, and no lecture on Tuesday.

Today's Agenda

- Fourier Transform Pitfalls
  - 2π factors
- Sampling Pitfalls
  - Impulses in the frequency domain
  - Do we really have to sample at the Nyquist rate?

1 Fourier Transform Pitfalls

1.1 2π factors

We've seen 2π and $1/2\pi$ factors appear all over the place in Fourier transform formulae and got headaches trying to remember them. They all stem from the fact that we use *angular* frequency ω instead of *cyclic* frequency, f, where $\omega = 2\pi f$. We view angular frequency as being more "natural," but many practical problems use cyclic frequency, so we need to remember when to add in factors of 2π. With this convention, we saw that the synthesis and analysis equations for the CT and DT Fourier transforms become:

$$\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega && \text{(CT synthesis, inverse CTFT)}\\
X(j\omega) &= \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt && \text{(CT analysis, CTFT)}\\
x[n] &= \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega && \text{(DT synthesis, inverse DTFT)}\\
X(e^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} && \text{(DT analysis, DTFT)}
\end{aligned}$$

The 2π factor is manifested in the following FT pairs and properties:

- **Value of a signal at zero time**

$$\begin{aligned}
x(0) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\,d\omega\\
x[0] &= \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\,d\omega
\end{aligned}$$

- **Constant signal**

$$\begin{aligned}
x(t) = 1 \quad &\overset{\mathcal{F}}{\longleftrightarrow} \quad X(j\omega) = 2\pi\delta(\omega)\\
x[n] = 1 \quad &\overset{\mathcal{F}}{\longleftrightarrow} \quad X(e^{j\omega}) = 2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega - 2\pi l)
\end{aligned}$$

- **Complex exponentials**

$$\begin{aligned}
x(t) = e^{j\omega_0 t} \quad &\overset{\mathcal{F}}{\longleftrightarrow} \quad X(j\omega) = 2\pi\delta(\omega - \omega_0)\\
x[n] = e^{j\omega_0 n} \quad &\overset{\mathcal{F}}{\longleftrightarrow} \quad X(e^{j\omega}) = 2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega - \omega_0 - 2\pi l)
\end{aligned}$$

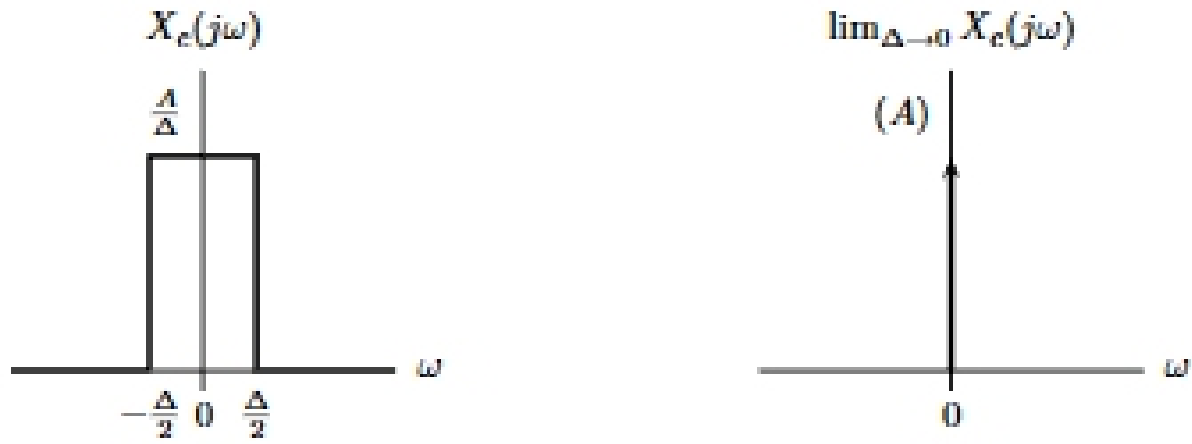
- **Multiplication property**

$$\begin{aligned}
r(t) = s(t)p(t) \quad &\overset{\mathcal{F}}{\longleftrightarrow} \quad R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega-\theta))\,d\theta = \frac{1}{2\pi}\left\{S(j\theta) * P(j\theta)\right\}\\
r[n] = s[n]p[n] \quad &\overset{\mathcal{F}}{\longleftrightarrow} \quad R(j\omega) = \frac{1}{2\pi}\int_{2\pi} S(e^{j\theta})P(e^{j(\omega-\theta)})\,d\theta = \frac{1}{2\pi}\left\{S(e^{j\theta}) \circledast P(e^{j\theta})\right\}
\end{aligned}$$

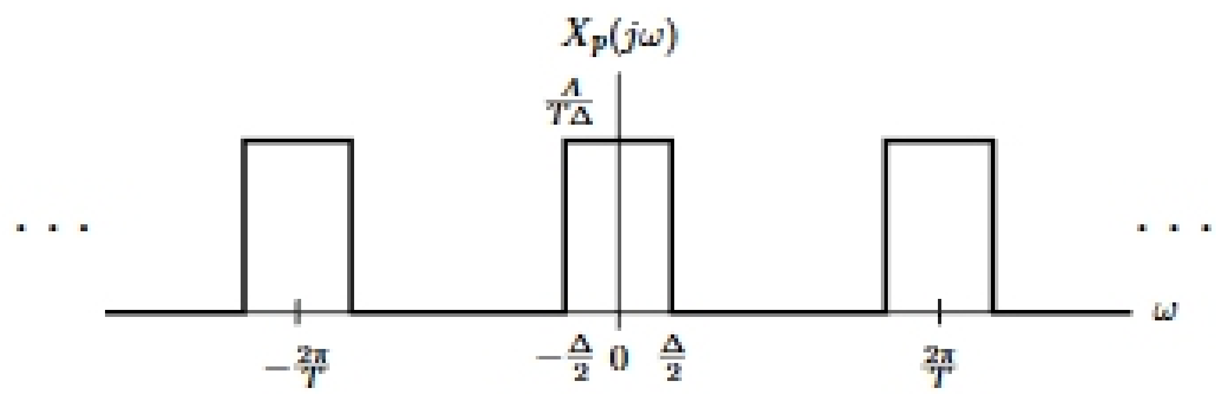
2 Sampling Pitfalls

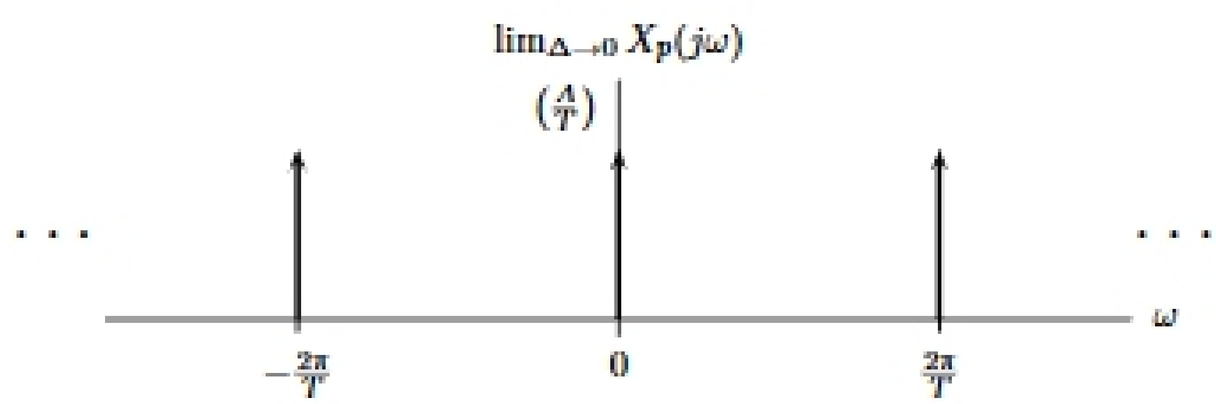
2.1 Impulses in the frequency domain

We need to be careful when there are impulses in the frequency domain. Let's consider the simplest case where $X_c(j\omega) = A\delta(\omega)$ (so $x_c(t) = A/(2\pi)$) and we sample $x_c(t)$ with period T. Recall that we label impulses by the *area* obtained when they are integrated. I shall follow the convention of surrounding that value with a pair of parentheses to remind us of this. To aid us in the process, let's consider the impulse $A\delta(\omega)$ to be the limit as $\Delta \to 0$ of a rectangular pulse with width Δ and height A/Δ. For our purposes here, it does not matter how this pulse is centered:



Thus, $X_p(j\omega)$ has the height $A/(T\Delta)$, retains the width Δ, and is replicated; the area is now $A/(T\Delta)\cdot\Delta = A/T$:





Finally, $X_d(e^{j\Omega})$ retains the height $A/(T\Delta)$, but its width is $T\Delta$; the area is now $A/(T\Delta)\cdot T\Delta = A$: