Lecture #16: The Frozen-in Flux Theorem

I. Review: Basic Concepts in MHD

A. MHD Equations

Continuity Eq. $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{U}) = 0$

Momentum Eq. $\rho \frac{\partial \underset{\sim}{U}}{\partial t} + \rho \underset{\sim}{U} \cdot \nabla \underset{\sim}{U} = -\nabla \left(p + \frac{B^2}{2\mu_0}\right) + \frac{(\underset{\sim}{B} \cdot \nabla)\underset{\sim}{B}}{\mu_0}$

Induction Eq. $\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B}) + \frac{\eta}{\mu_0} \nabla^2 \underset{\sim}{B}$

Energy Eq. (Adiabatic) $\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0$

1. Resistive MHD when $\eta \neq 0$
2. Ideal MHD when $\eta = 0$

B. MHD Approximation

1. Strong Collisions: $\lambda_m \ll L$, $\tau \gg \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} \nu_{ei}^{-1}$
2. Non-relativistic: $v_0 \ll c$
3. Magnetized: $r_{Li} \ll L$

C. Properties of MHD

1. Quasineutrality: $\sum_s n_s q_s \simeq 0$
2. Ohm's Law: $\underset{\sim}{E} + \underset{\sim}{U} \times \underset{\sim}{B} = \eta \underset{\sim}{J}$

   a. Typically, the conductivity of plasmas is very high, so η is small.

   b. Usually, $|\eta \underset{\sim}{J}| \ll |\underset{\sim}{U} \times \underset{\sim}{B}|$, so $\underset{\sim}{E} \simeq -\underset{\sim}{U} \times \underset{\sim}{B}$

   Thus, the fluid velocity $\underset{\sim}{U}$ corresponds to the $\underset{\sim}{E} \times \underset{\sim}{B}$ velocity

II. Dimensionless Numbers in Fluid Dynamics

A. General

1. In fluid dynamics, a common way to characterize the behavior of fluids in different physical systems is to calculate dimensionless numbers characteristic of the flow.
2. These dimensionless numbers typically characterize the magnitude of the ratio of two terms in the dynamical equations.

B. Example: Reynolds Number in Hydrodynamics, Re

1. Navier-Stokes Equation: $\rho \frac{\partial \underset{\sim}{U}}{\partial t} + \rho \underset{\sim}{U} \cdot \nabla \underset{\sim}{U} = -\nabla p + \mu \nabla^2 \underset{\sim}{U}$

   where μ is the coefficient of shear viscosity.

2. We can divide by the density to yield:

$$\frac{\partial \underset{\sim}{U}}{\partial t} + \overbrace{\underset{\sim}{U} \cdot \nabla \underset{\sim}{U}}^{\text{convection}} = -\frac{1}{\rho} \nabla p + \overbrace{\nu \nabla^2 \underset{\sim}{U}}^{\text{diffusion}}$$

   where we define $\boxed{\text{Kinematic Viscosity } \nu \equiv \frac{\mu}{\rho}}$

3. The Reynolds Number is defined as ratio of convection to diffusion term.

   a. $$Re \equiv \frac{|\underset{\sim}{U} \cdot \nabla \underset{\sim}{U}|}{|\nu \nabla^2 \underset{\sim}{U}|} \sim \frac{(V_0^2/L)}{\nu V_0/L^2} \sim \frac{L V_0}{\nu}$$

   b. Thus $\boxed{Re \equiv \frac{L V_0}{\nu}}$

4. Low Re flows ($Re <$ few hundred) $\Rightarrow$ laminar

   High Re flows ($Re \gtrsim 10^3$) $\Rightarrow$ turbulent.

C. Magnetic Reynolds Number: Re_M

1. Induction Equation: $$\frac{\partial \underset{\sim}{B}}{\partial t} = \underbrace{\nabla \times (\underset{\sim}{U} \times \underset{\sim}{B})}_{\text{convection}} + \underbrace{\frac{\eta}{\mu_0} \nabla^2 \underset{\sim}{B}}_{\text{diffusion}}$$

II. (Continued)

2. The <u>Magnetic Reynolds Number</u> is the ratio of convection to diffusion term in the induction equation,

$$Re_M \equiv \frac{|\nabla\times(\underset{\sim}{U}\times\underset{\sim}{B})|}{\left|\frac{\eta}{\mu_0}\nabla^2\underset{\sim}{B}\right|} \sim \frac{\left(\frac{V_0 B}{L}\right)}{\frac{\eta B}{\mu_0 L^2}} \sim \frac{\mu_0 L V_0}{\eta}$$

Thus $\boxed{Re_M \equiv \frac{\mu_0 L V_0}{\eta}}$

3. a. In the limit $Re_M \gg 1$, the convection term dominates and diffusion can be ignored $\Rightarrow$ Ideal MHD!

b. In the limit $Re_M \ll 1$, the diffusion term dominates.

III. <u>The Frozen-in Flux Theorem:</u>

Most plasmas satisfy the condition $Re_M \gg 1$, giving

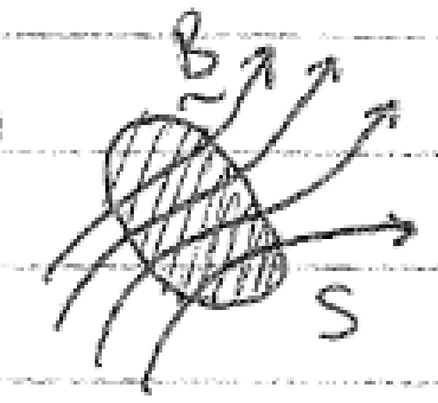
$$\frac{\partial \underset{\sim}{B}}{\partial t} + \nabla\times(\underset{\sim}{U}\times\underset{\sim}{B}) = 0 \qquad \text{Ideal MHD Induction Eq.}$$

In this limit, a powerful theorem can be proven.

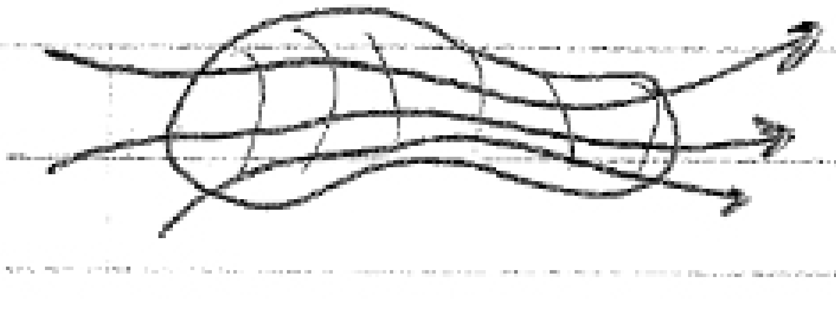
A. <u>The Frozen-in Flux Theorem:</u> The Magnetic flux through a surface moving with the plasma (at fluid velocity $\underset{\sim}{U}(\underset{\sim}{r},t)$) remains constant.

<u>Proof:</u>

1. First, we define Magnetic Flux $\boxed{\Phi_B = \int_S \underset{\sim}{B}\cdot d\underset{\sim}{A}}$



2.a. <u>Fact:</u> The flux through any closed surface is zero.



$$\oint_S \underset{\sim}{B}\cdot d\underset{\sim}{A} = \int_V \nabla\cdot\underset{\sim}{B}\, d^3\underset{\sim}{r} = 0$$

Divergence Theorem, NRL p.5 (28)