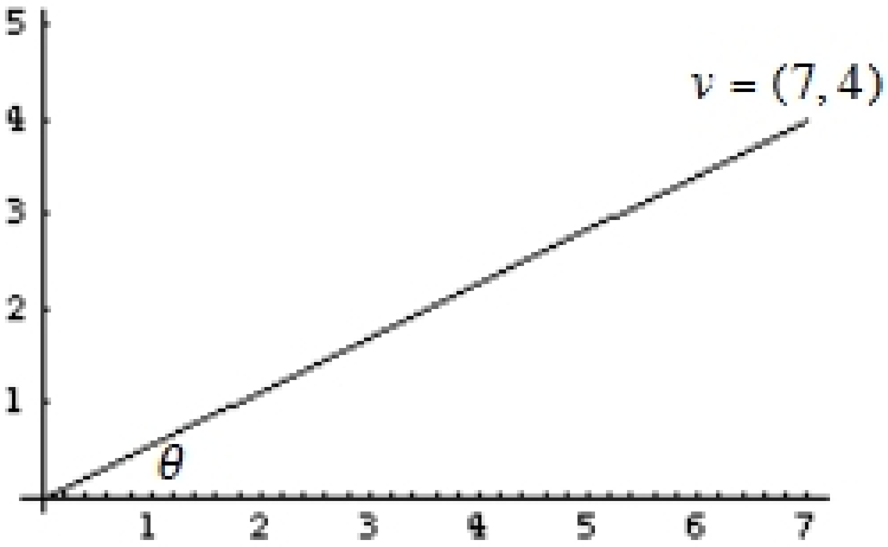
MATH 117 Vectors and Polar Form

Any point (x, y) in the xy plane forms a directed line segment *from* the origin (0, 0) *to* the point (x, y). Such a segment is called a *vector*. When we want to consider the vector and not just the point, then we generally label it as $v = (x, y)$.



Length and Direction

A vector $v = (x, y)$ has a length (or norm) denoted by $\|v\|$ which is simply the distance to the origin given by the hypotenuse. The direction θ is the standard angle determined by (x, y) as measured from the positive x axis. Thus we have

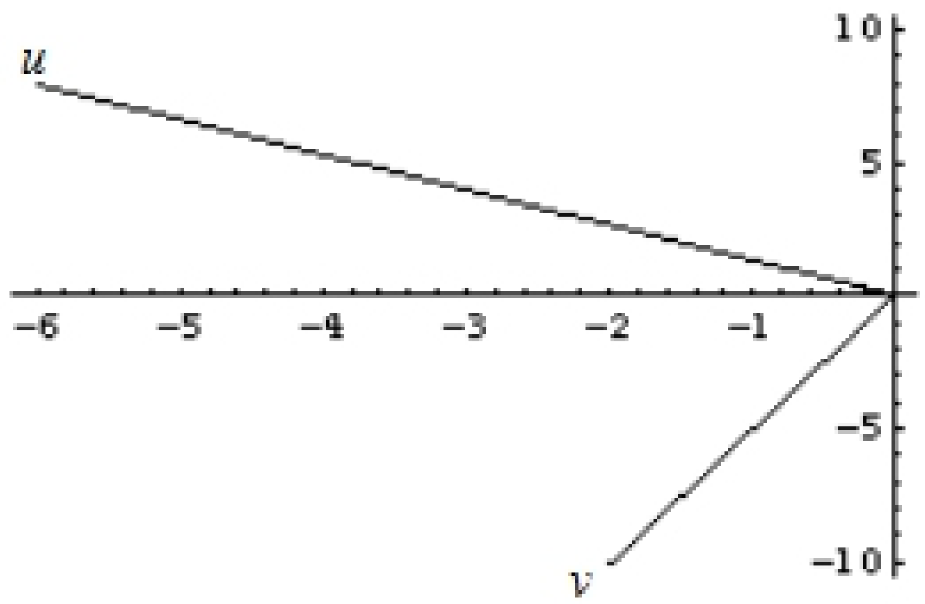
$$\|v\| = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \tan\theta = \frac{y}{x}$$

To find the direction θ, compute $\tan^{-1}(y/x)$ and then adjust the angle to the proper quadrant. A vector written in terms of its length and direction is then in *polar form*.

Example 1. Let $u = (-6, 8)$ and $v = (-2, -10)$. Find the length and direction of each vector and write the vectors in polar form.

Solution. Vector u has length $\|u\| = \sqrt{6^2 + 8^2} = 10$. Here, $\tan^{-1}(y/x)$ is $\tan^{-1}(8/-6) \approx -53.13°$. So in Quad. II, $\theta = -53.13° + 180° \approx 126.87°$.

So $u = (10, 126.87°)$ in polar form.



Vector v has length $\|v\| = \sqrt{2^2 + 10^2} = \sqrt{104}$. Its angle in Quadrant III is given by $\theta = \tan^{-1}(10/2) + 180° \approx 258.69°$. Then $v = (\sqrt{104}, 258.69°)$ in polar form.

Converting Polar Form Back to Rectangular Form

If a vector is given in polar form ($\|v\|$, θ), then we recover the x and y coordinates by

$$x = \|v\| \cos\theta \quad \text{and} \quad y = \|v\| \sin\theta$$

Here, $\|v\|$ is taking the place of the radius r, where $x = r\cos\theta$ and $y = r\sin\theta$.

***Example* 2.** Find the rectangular form of the vectors u = (30, 120°) and v = (20, 330°).

Solution. For u = (30, 120°), we have $x = 30\cos 120° = 30 \times \left(-\frac{1}{2}\right) = -15$ and $y = 30\sin 120° = 30 \times \left(\frac{\sqrt{3}}{2}\right) = 15\sqrt{3}$; so $u = (-15, 15\sqrt{3})$.

For v, $x = 20\cos 330° = 20 \times \left(\frac{\sqrt{3}}{2}\right) = 10\sqrt{3}$ and $y = 20\sin 330° = 20 \times \left(-\frac{1}{2}\right) = -10$; so then $v = (10\sqrt{3}, -10)$.

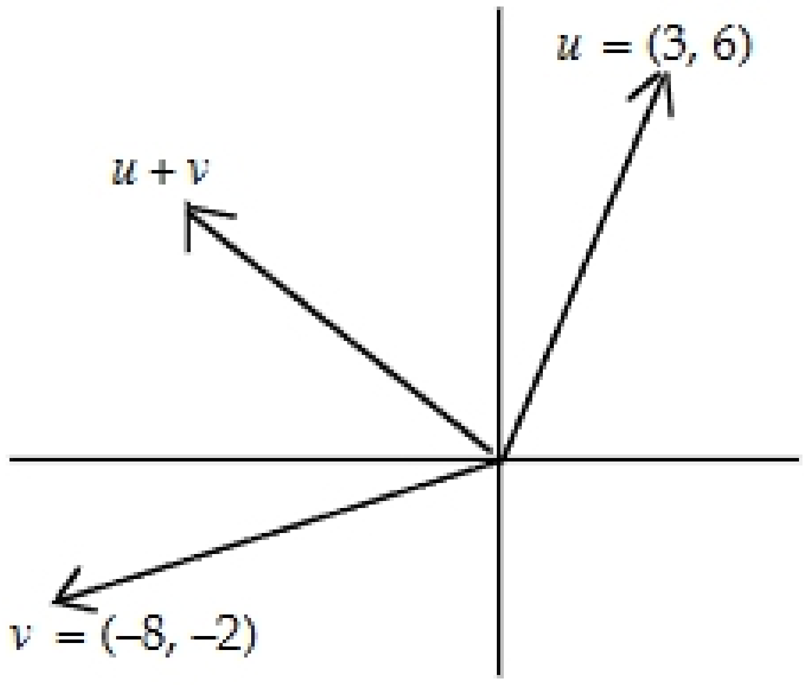
Adding Vectors

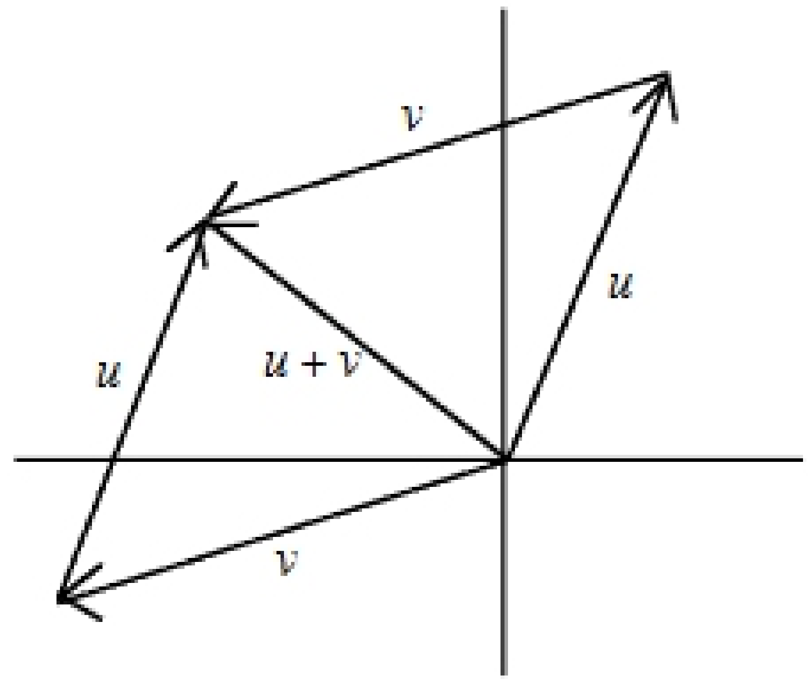
Given two vectors $u = (x_1, y_1)$ and $v = (x_2, y_2)$, both in rectangular form, we obtain the sum of vectors by adding component wise: $u + v = (x_1 + x_2, y_1 + y_2)$. If we make a parallelogram out of the vectors u and v, then vector $u + v$ is the diagonal that starts at the origin.

***Example* 3.** Let u = (3, 6) and v = (–8, –2). Graph u, v, and $u + v$. What is the length and direction of $u + v$?

Solution. First, $u + v = (3 + (-8), 6 + (-2)) = (-5, 4)$, which we see to be the diagonal of the parallelogram determined by u and v. (See graphs on next page.)

Then $\|u + v\| = \sqrt{5^2 + 4^2} = \sqrt{41} \approx \mathbf{6.4}$, and the direction of $u + v$ is given by $\theta = \tan^{-1}(-4/5) + 180° \approx \mathbf{141.34°}$.





Adding Forces

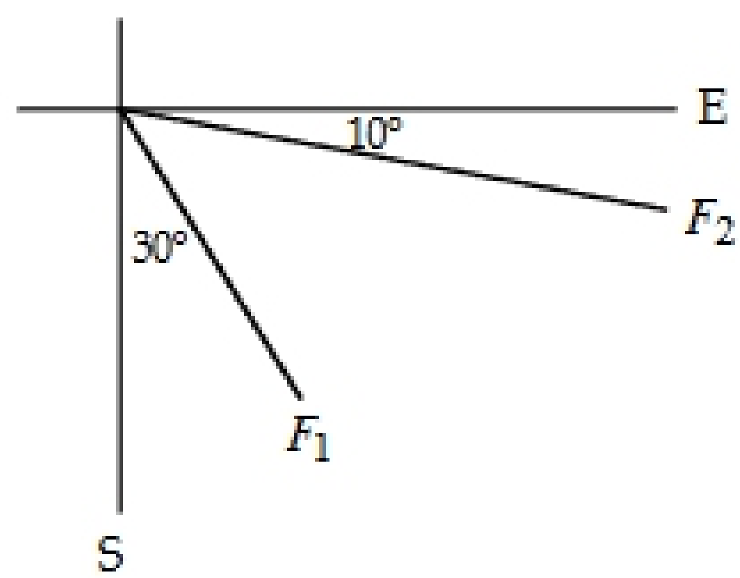
Often, a force is given in terms of its magnitude and direction. In order to add two forces, we convert each to rectangular form, add the x and y components to get the sum, then convert the result back to polar form. The sum of two forces is called the *resultant force*.

Example 4. Let F_1 be a force of 50 Newtons in the direction 30° East of South, and let F_2 be a force of 80 Newtons in the direction 10° South of East. Find the magnitude and direction of the resultant $F_1 + F_2$.

Solution. First, the angle for F_1 is 30° + 270° = 300°, and the angle for F_2 is 360° – 10° = 350°. Next, the x and y components for each force and the resultant are:

$$F_1 = (50\cos 300°,\ 50\sin 300°)$$

$$F_2 = (80\cos 350°,\ 80\sin 350°)$$



So the resultant force is

$$F_1 + F_2 = (50\cos 300° + 80\cos 350°,\ 50\sin 300° + 80\sin 350°) = \mathbf{(103.78462, -57.19312)}$$

So $F_1 + F_2$ has magnitude $|F_1 + F_2| = \sqrt{103.78462^2 + 57.19312^2} \approx$ **118.5 Newtons**. Its direction is in the 4th Quadrant is $\tan^{-1}(-57.19312/103.78462) + 360° \approx$ **331.142°**, or about 28.858° South of East.