```
> restart;
```

Fourier Polynomials and Series

0.1 Fourier Polynomials

Fourier polynomials provide a way of approximating general periodic functions by sums of very simple periodic functions, namely the familiar sine and cosine functions, shifted and scaled. We will consider functions, f(x), that are 2π -periodic, so that f(x $+2\pi$) = f(x). For example, any function of the form$f(x) = b_k \sin(kx)$ or$g(x) = a_k \cos(kx)$, where k is a positive integer whilea_k andb_k are constants, is 2π -periodic, although it may have a smaller*period*.That is, the period of , say,$\sin(kx)$ is$2\frac{\pi}{k}$, so that it repeats after every interval of length$2\frac{\pi}{k}$, but then of course it repeats after every interval of length 2π . The integer k gives the*frequency*of the sine or cosine function, so large values of k correspond to very wiggly graphs. The numbersa_k andb_k give the*amplitudes*. Recall also that the graph of cos(x) is obtained from the graph of sin(x) by a horizontal shift ofπ /2, and the graph of Acos(kx) is likewise obtained from the graph of Asin(kx) by a horizontal shift of lengthπ /(2k). What do you get if you shift the graph of Asin(kx) horizontally by some other distance? It turns out that any such horizontal shift of Asin(kx) can be represented in the form

(1)$a_k \cos(kx) + b_k \sin(kx)$

where the amplitude A is equal to$\sqrt{{a_k}^2 + {b_k}^2}$. Functions of the form (1) are convenient to work with. Sums of such functions, with possibly an added constant which we denote by , are called*Fourier polynomials*.Thus a Fourier polynomial of*degree n*is a function of the form F(x) =

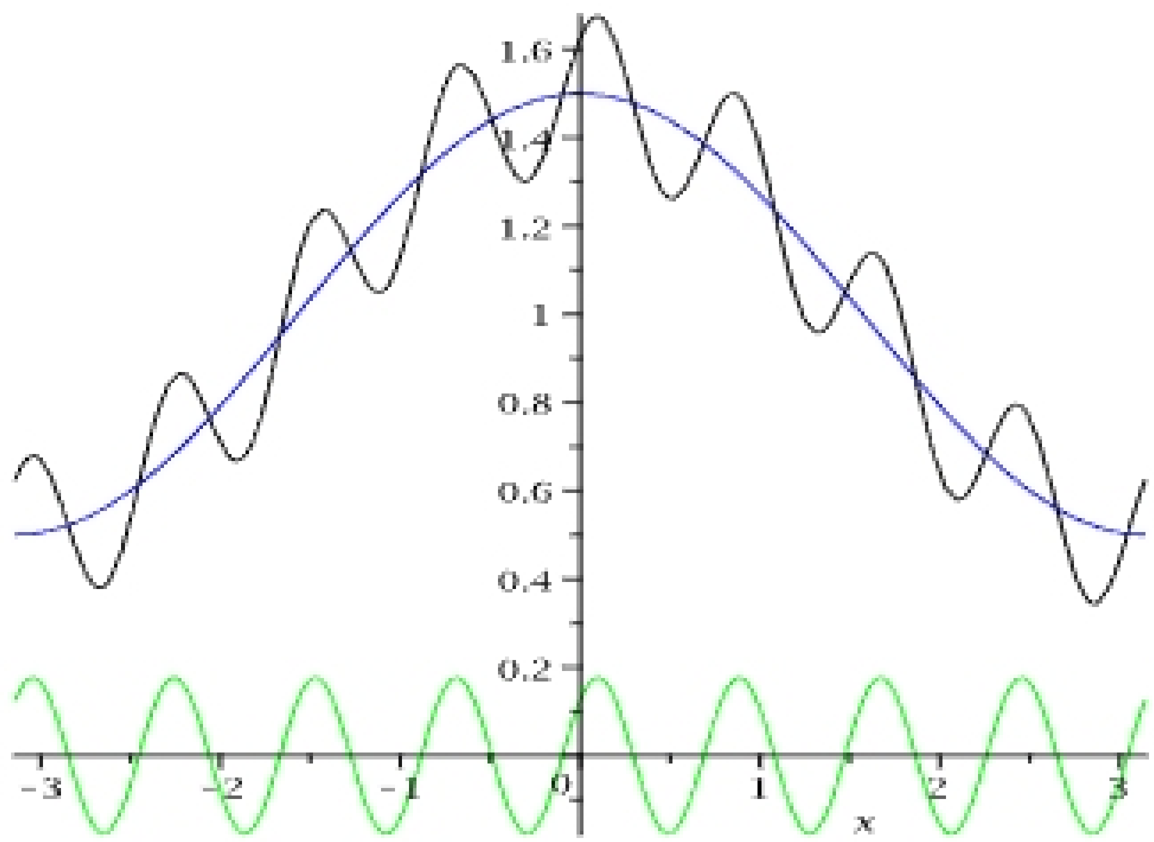
$$a_0 + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx) = a_0 + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx)$$

.

A Fourier polynomial can be specified by giving the*Fourier coefficients* , ... For example, suppose$a_0 = 1$, $a_1 = 1/2$, $a_8 = 1/8$, $b_8 = 1/8$, with all other coefficients being 0. The corresponding Fourier polynomials is then

F(x) =$1 + 1/2\cos(x) + 1/8\cos(8x) + 1/8\sin(8x)$. Let's graph the three functions$1 + 1/2\cos(x)$,$1/8\cos(8x) + 1/8\sin(8x)$, and F(x) on the same axes:

```
> plot([1+cos(x)/2,cos(8*x)/8+sin(8*x)/8,1+cos(x)/2+cos(8*x)/8+sin(8*x)/8],
x=-Pi..Pi, color=[blue, green,black]);
```



Note that we only graphed between $-\pi$ and π , a practice we will continue throughout the worksheet. Observe that the high frequency term is a shifted sine with amplitude $\sqrt{1/32}$ ˜ 0.177 and that the graph of F(x) is the superposition of this high frequency graph with that of the vertically shifted cos(x)/2.

It will be convenient to specify the Fourier coefficients by giving the constant term , and then two*lists*for the coefficents$a_1 \ldots a_n$, and$b_1 \ldots b_n$. The procedureFPolydefined below, takes the constant term , the two lists and the degree and returns the Fourier polynomial as an expression. For our example, the coefficient list for the a's is [1/2,0,0,0,0,0,0,0,0,0,1/8] while the coefficient list for the b's is [0,0,0,0,0,0,0,0,0,0,1/8].Be sure to execute the next command line.

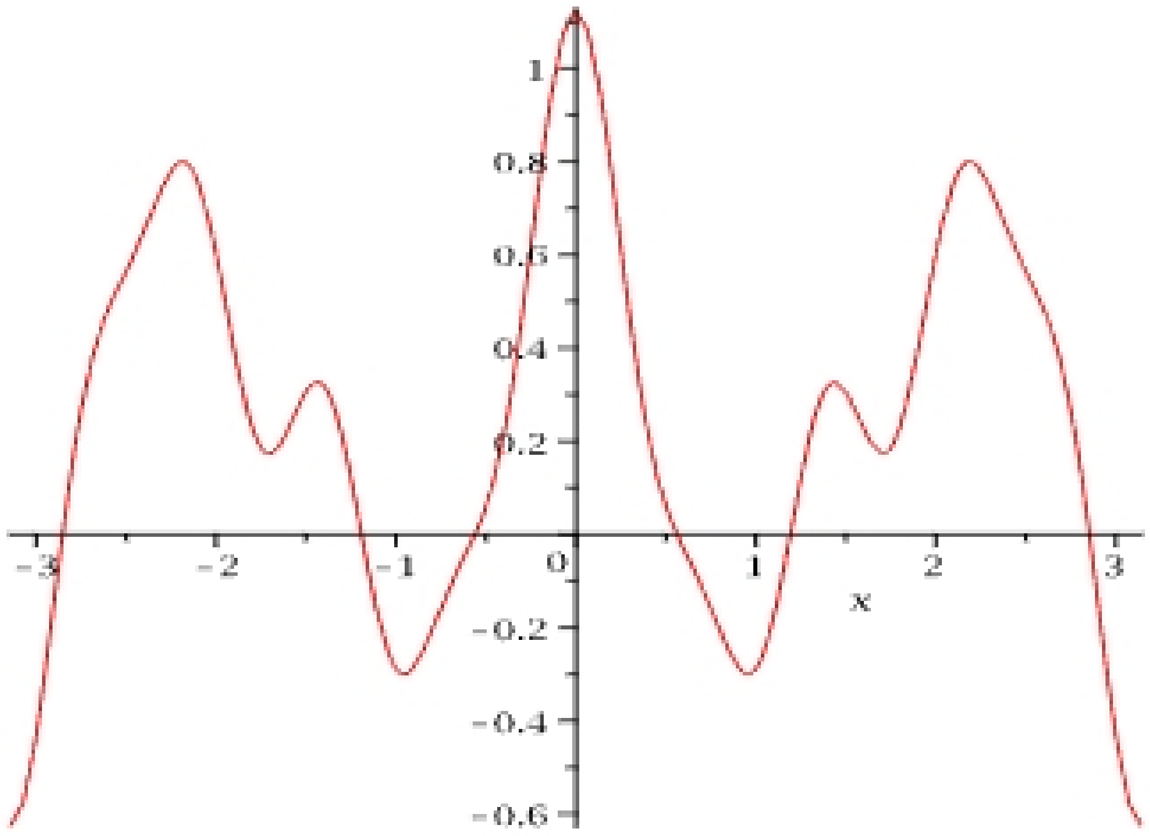
```
> FPoly:=proc(a0,a,b,n) local k;
a0+sum('a[k]*cos(k*x)+b[k]*sin(k*x)','k'=1..n); end;
```

$FPoly :=$ **proc**$(a0abn)$**local** k; $a0 + sum('a[k]*cos(k*x) + b[k]*sin(k*x)', 'k' = 1..n)$**end proc**;

As an example let's plot the degree 9 Fourier polynomial F(x) =$0.25 + 0.5\cos(3x) + 0.25\cos(5x) + 0.125\cos(9$

.

```
> plot(
FPoly(.25, [0,0,.5,0,.25,0,0,0,.125],[0,0,0,0,0,0,0,0,0],9),x=-Pi..Pi);
```



Note that this is the graph of an *even* function of x, and that all the sine coefficients were 0. This is an example of the following general fact you will find useful.

Odd and Even:

A Fourier polynomial is an even function when all its sine coefficients are 0. A Fourier polynomial with its constant term removed is an odd function when all the cosine coefficients are 0.

Problem 1. By trial and error, estimate the Fourier coefficients of the degree 4 Fourier polynomials whose graphs are shown below. All plots are from $-\pi$ to π and (hint) each has no more than 3 nonzero coefficients.

a)

b)

c)

0.2 The Fourier Polynomials of a Function

If f(x) is a 2π periodic function, we define the *Fourier coefficients of f* to be

$$a_0 = 1/2\,\frac{\int_{-\pi}^{\pi} f(x)\,dx}{\pi},\quad a_k = \frac{\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx}{\pi},\quad b_k = \frac{\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx}{\pi}$$

, for k = 1,2,...n.

The degree n Fourier polynomial with these coefficients is called the *degree n Fourier polynomial of f*. The infinite series whose partial sums are the Fourier