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16.07 Dynamics
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Lecture L13 - Conservative Internal Forces and Potential Energy

The forces internal to a system are of two types. Conservative forces, such as gravity; and dissipative forces such as friction. Internal forces arise from the natural dynamics of the system in contract to external forces which are imposed from an external source.

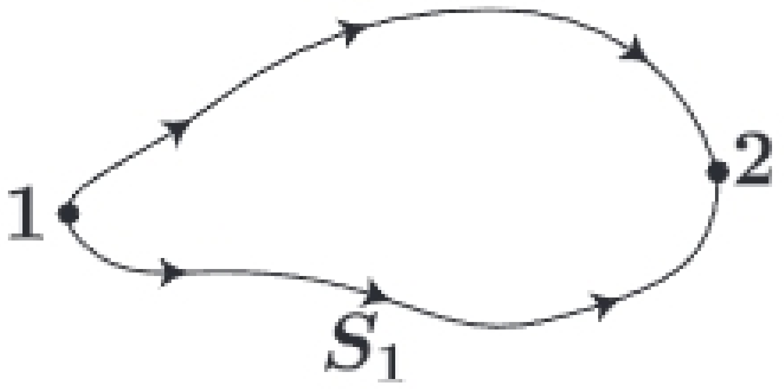
We have seen that the work done by a force $\boldsymbol{F}$ on a particle is given by $dW = \boldsymbol{F} \cdot d\boldsymbol{r}$.

If the work done by an internal forces $\boldsymbol{F}$, when the particle moves from any position $\boldsymbol{r}_1$ to any position $\boldsymbol{r}_2$, can be expressed as the difference in a scalar function of $\boldsymbol{r}$ between the two ends of the trajectory,

$$W_{12} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{r} = -(V(\boldsymbol{r}_2) - V(\boldsymbol{r}_1)) = V_1 - V_2 \ , \tag{1}$$

then we say that the force is *conservative*. In the above expression, the scalar function $V(\boldsymbol{r})$ is called the *potential*. It is clear that the potential satisfies $dV = -\boldsymbol{F} \cdot d\boldsymbol{r}$ (the minus sign is included for convenience).

There are two main consequences that follow from the existence of a potential: i) the work done by a conservative force between points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ is *independent of the path*. This follows from (1) since W_{12} only depends on the initial and final potentials V_1 and V_2 (and not on how we go from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$), and ii) the work done by potential forces is *recoverable*. Consider the work done in going from point $\boldsymbol{r}_1$ to point $\boldsymbol{r}_2$, W_{12}. If we go, now, from point $\boldsymbol{r}_2$ to $\boldsymbol{r}_1$, we have that $W_{21} = -W_{12}$ since the total work $W_{12} + W_{21} = (V_1 - V_2) + (V_2 - V_1) = 0$.



In one dimension any force which is only a function of position is conservative. That is, if we have a force, $F(x)$, which is only a function of position, then $F(x)\,dx$ is always a perfect differential. This means that we can define a potential function as

$$V(x) = -\int_{x_0}^{x} F(x)\,dx \ ,$$

where x_0 is arbitrary.

In two and three dimensions, we would, in principle, expect that any force which depends only on position, $\boldsymbol{F}(\boldsymbol{r})$, to be conservative. However, it turns out that, in general, this is not sufficient. In multiple dimensions, the condition for a force field to be conservative is that it can be expressed as the gradient of a potential function. That is,

$$\boldsymbol{F}_C = -\boldsymbol{\nabla} V \ .$$

This result follows from the gradient theorem, which is often called the fundamental theorem of calculus, which states that the integral

$$-\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{\nabla} V \cdot d\boldsymbol{r} = -(V_2 - V_1)$$

is independent of the path between $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. Therefore the work done by conservative forces depends only upon the endpoints $\boldsymbol{r}_2$ and $\boldsymbol{r}_1$ rather than the details of the path taken between them.

$$\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}_C \cdot d\boldsymbol{r} = -\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{\nabla} V \cdot d\boldsymbol{r} = -(V_2 - V_1)$$

In the general case, we will deal with internal forces that are a combination of conservative and non-conservative forces.

$$\boldsymbol{F} = \boldsymbol{F}_C + \boldsymbol{F}_{NC} = -\boldsymbol{\nabla} V + \boldsymbol{F}_{NC} \ .$$

Note ***The gradient operator, $\boldsymbol{\nabla}$***

The gradient operator, $\boldsymbol{\nabla}$ (called "del"), in cartesian coordinates is defined as

$$\boldsymbol{\nabla}(\ ) \equiv \frac{\partial(\ )}{\partial x}\boldsymbol{i} + \frac{\partial(\ )}{\partial y}\boldsymbol{j} + \frac{\partial(\ )}{\partial z}\boldsymbol{k} \ .$$

When operating on a scalar function $V(x, y, z)$, the result $\boldsymbol{\nabla} V$ is a vector, called the gradient of V. The components of $\boldsymbol{\nabla} V$ are the derivatives of V along each of the coordinate directions,

$$\boldsymbol{\nabla} V = \frac{\partial V}{\partial x}\boldsymbol{i} + \frac{\partial V}{\partial y}\boldsymbol{j} + \frac{\partial V}{\partial z}\boldsymbol{k} \ .$$

If we consider a particle moving due to conservative forces with potential energy $V(x, y, z)$, as the particle moves from point $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ to point $\boldsymbol{r} + d\boldsymbol{r} = (x + dx)\boldsymbol{i} + (y + dy)\boldsymbol{j} + (z + dz)\boldsymbol{k}$, the potential energy changes by $dV = V(x + dx, y + dy, z + dz) - V(x, y, z)$. For small increments dx, dy and dz, and dV, can be expressed, using Taylor series expansions, as

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = \boldsymbol{\nabla} V \cdot d\boldsymbol{r} \ ,$$

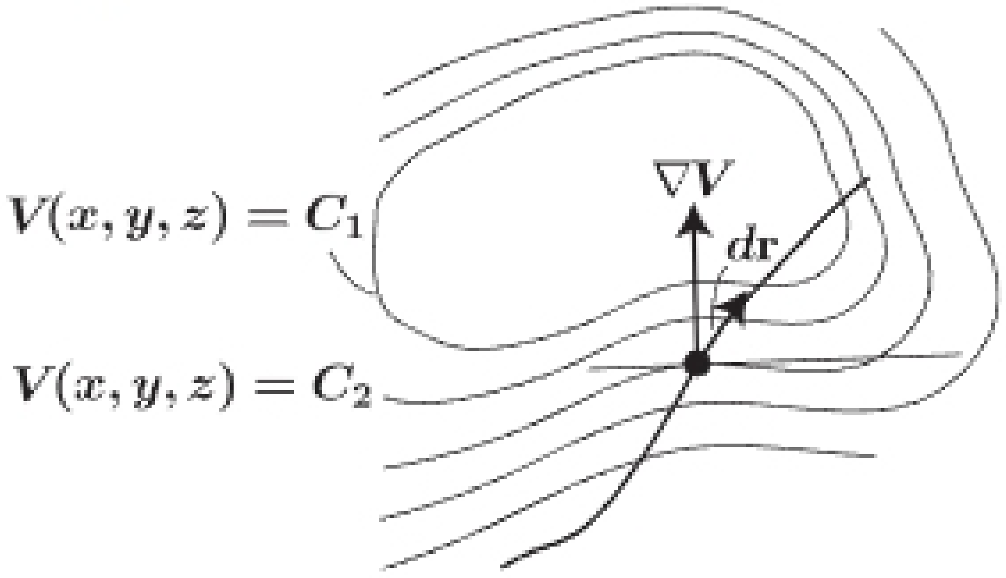
where $d\boldsymbol{r} = dx\boldsymbol{i} + dy\boldsymbol{j} + dz\boldsymbol{k}$.

This equation expresses the fundamental property of the gradient. The gradient allows us to find the change in a function induced by a change in its variables.

If we write $V(x, y, z) = C$, for some constant C, this is the implicit equation of a surface, which is called a constant energy surface. This surface is made up by all the points in the x, y, z space for which the function $V(x, y, z)$ is equal to C. It is clear that if a particle moves on a constant energy surface, $dV = 0$, since V is constant on that surface. Therefore, when a particle moves on a constant energy surface, $\boldsymbol{dr}$ will be tangent to that surface, and since

$$0 = dV = \boldsymbol{\nabla} V \cdot \boldsymbol{dr} \ ,$$

we have that $\boldsymbol{\nabla} V$ is perpendicular to any tangent to the surface. This situation is illustrated in the picture below for the two dimensional case. Here, the constant energy surfaces are contour curves, and we can see that the gradient vector is always normal to the contour curves.



Note ***Gradient operator in cylindrical coordinates***

The gradient operator can be expressed in cylindrical coordinates by writing $x = r\cos\theta$, $y = r\sin\theta$, and $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$. Thus, applying the chain rule for differentiation, we have

$$\begin{aligned} \frac{\partial(\ )}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial(\ )}{\partial r} + \frac{\partial r}{\partial x}\frac{\partial(\ )}{\partial \theta} = \cos\theta\frac{\partial(\ )}{\partial r} - \frac{\sin\theta}{r}\frac{\partial(\ )}{\partial r} \\ \frac{\partial(\ )}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial(\ )}{\partial r} + \frac{\partial r}{\partial y}\frac{\partial(\ )}{\partial \theta} = \sin\theta\frac{\partial(\ )}{\partial r} + \frac{\cos\theta}{r}\frac{\partial(\ )}{\partial r} \ . \end{aligned}$$

If we note that $\boldsymbol{i} = \cos\theta\boldsymbol{e}_r - \sin\theta\boldsymbol{e}_\theta$ and $\boldsymbol{j} = \sin\theta\boldsymbol{e}_r + \cos\theta\boldsymbol{e}_\theta$, we have that

$$\boldsymbol{\nabla}(\ ) \equiv \frac{\partial(\ )}{\partial r}\,\boldsymbol{e}_r + \frac{1}{r}\frac{\partial(\ )}{\partial \theta}\,\boldsymbol{e}_\theta + \frac{\partial(\ )}{\partial z} \ .$$

An expression for spherical coordinates can be derived in a similar manner.

Conservation of Energy

When all the forces doing work are conservative, the work is given by (1), and the principle of work and energy derived in the last lecture,

$$T_1 + W_{12} = T_2 \ ,$$

reduces to,

$$T_1 + V_1 = T_2 + V_2$$