Consider the coordinate transformation defined by $x = u + \frac{v}{2}$ and $y = v$.

A. Solve the system for u and v in terms of x and y.

B. Find the value of the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$.

C. Find the image of the region $\left\{(x,y) \,\middle|\, \frac{y}{2} \le x \le \frac{y+4}{2} \text{ and } 0 \le y \le 2\right\}$ under the transformation.

D. Use a change of variables to evaluate the integral $\int_0^2 \int_{y/2}^{(y+4)/2} y^3 (2x - y) e^{(2x-y)^2} \, dx \, dy$.

Practice Exam 3

November 17, 2023 **Solutions to Work-out problems**

1. Consider the coordinate transformation defined by $x = u + \frac{v}{2}$ and $y = v$.
   (a) Solve the system for u and v in terms of x and y.
   (b) Find the value of the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$.
   (c) Find the image of the region

$$\left\{(x,y)\middle|\frac{y}{2} \le x \le \frac{y+4}{2} \text{ and } 0 \le y \le 2\right\}$$

   under the transformation.
   (d) Use a change of variables to evaluate the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2}\,dx\,dy$$

(a) The start of this is pretty easy. We know that $v = y$. Afterwards, we can replace v in the equation for x to get $x = u + \frac{y}{2} \implies u = x - \frac{y}{2}$.

(b) We compute

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{bmatrix} = \det\begin{bmatrix}1 & 1/2\\ 0 & 1\end{bmatrix} = 1.$$

$(1\times 1)+(\frac{1}{2}\times 0) = 1$

(c) For this part, we change each of the components of the set that is given.

$$x = u + \frac{v}{2} \qquad \frac{y}{2} = \frac{v}{2} \qquad \frac{y+4}{2} = \frac{y}{2} + 2 = \frac{v}{2} + 2$$

So the first inequality gives

$$\frac{v}{2} \le u + \frac{v}{2} \le \frac{v}{2} + 2 \implies 0 \le u \le 2.$$

Replacing y with v in the second inequality gives $0 \le v \le 2$. Then we have a square

$$\{(u,v)|0 \le u \le 2 \text{ and } 0 \le v \le 2\}.$$

(d) We first start with the integrand. In particular, notice that $u = x - \frac{y}{2} \implies 2u = 2x - y$, and therefore

$$y^3(2x-y)e^{(2x-y)^2} = v^3(2u)e^{(2u^2)} = 2v^3ue^{4u^2}.$$

So we have

For calculator

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2}\,dx\,dy = \int_0^2 \int_0^2 2v^3ue^{4u^2}\,du\,dv = \left(\int_0^2 v^3\,dv\right)\left(\int_0^2 2ue^{4u^2}\,du\right)$$

Now we only need to compute each integral individually. First, $\int_0^2 v^3\,dv = \frac{v^4}{4}\Big|_0^2 = 4$.
For the other integral, it's useful to make the substitution

$$s = 4u^2 \implies \begin{matrix} ds = 8u\,du \implies \frac{1}{4}ds = 2u\,du \\ s(0) = 0 \quad \text{and} \quad s(2) = 16 \end{matrix}$$

So we compute

$$\int_0^2 2ue^{4u^2}\,du = \frac{1}{4}\int_0^{16} e^s\,ds = \frac{1}{4}e^s|_0^{16} = \frac{1}{4}(e^{16}-1).$$

Combining all of these parts together, we have

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2}\,dx\,dy = e^{16} - 1.$$

Use Green's theorem to evaluate the line integral

$\oint_C (6y + x)\, dx + (y + 2x)\, dy$ **where** C **is the circle defined by** $(x-2)^2 + (y-3)^2 = 4$.

2. Use Green's theorem to evaluate the line integral

$$\oint_C (6y + x)\, dx + (y + 2x)\, dy \text{ where } C \text{ is the circle defined by } (x-2)^2 + (y-3)^2 = 4.$$

Let $\mathcal{O}$ be the region that is the circle C filled in. Then if $(M, N) = (6y + x, y + 2x)$, we have

$$\begin{aligned}\oint_C (6y + x)\, dx + (y + 2x)\, dy &= \iint_{\mathcal{O}} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) d\mathcal{O} \\ &= \iint_{\mathcal{O}} (2 - 6)\, d\mathcal{O} \\ &= -4 \cdot (\text{The area of } \mathcal{O})\end{aligned}$$

Since $\mathcal{O}$ is a circle of radius 2, it has area 4π. Therefore,

$$\oint_C (6y + x)\, dx + (y + 2x)\, dy = -16\pi.$$