Math 231E. Fall 2013. Midterm 1 Practice Solutions.

Problem 1. Compute the derivatives of the following functions

a. $f(x) = \sin(x^2)e^x$

b. $f(x) = \text{arcsec}(x)$

c. $f(x) = \ln(1 + x^2 + x^4)$

Solution:

a. We use the product and Chain Rules:

$$\frac{d}{dx}(\sin(x^2)e^x) = \frac{d}{dx}(\sin(x^2))e^x + \sin(x^2)\left(\frac{d}{dx}e^x\right) = 2x\cos(x^2)e^x + e^x\sin(x^2).$$

b. We have to use implicit methods here. We write $y = \sec^{-1}(x)$, *or*

$$x = \sec(y).$$

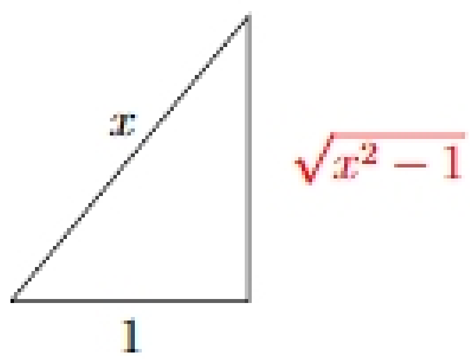
Differentiating both sides gives

$$1 = \sec(y)\tan(y)\frac{dy}{dx} = \frac{\sin(y)}{\cos^2(y)}\frac{dy}{dx},$$

so

$$\frac{dy}{dx} = \frac{\cos^2(y)}{\sin(y)}.$$

Draw a triangle with near side of length x *and far side of length* 1, *and if the interior angle is* y, *then* $\sec(y) = x$. *Since the hypotenuse of the triangle is* $\sqrt{1+x^2}$, *this means that*



$$\sin(y) = \frac{\sqrt{x^2-1}}{x}, \quad \cos(y) = \frac{1}{x},$$

so

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}.$$

c. *Using the Chain Rule,*

$$f'(x) = \frac{2x + 4x^3}{1 + x^2 + x^4}.$$

Problem 2. Give the Taylor polynomial for the following functions to the order indicated about the point $a = 0$

a. $f(x) = (\sin(x) - x)e^x$, order $= 4$

b. $f(x) = (\cos(x))/(1 - x)$, order $= 3$

c. $f(x) = \dfrac{1}{1 + x^2 + x^4}$, order $= 8$

d. $f(x) = x(\cos(x^2) - 1)\sin(x)$, order $= 10$

e. $f(x) = \ln(1 + x^2)$, order 6

Solution:

a. *We write out each one as follows:*

$$\sin(x) - x = -\frac{x^3}{6} + O(x^5),$$
$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4).$$

Multiplying these gives

$$(\sin(x) - x)e^x = \left(-\frac{x^3}{6} + O(x^5)\right)\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4)\right) = -\frac{x^3}{6} - \frac{x^4}{6} + O(x^5).$$

b. *We could do long division* ☹. *But we can be more clever. Recall that*

$$\frac{1}{x - 1} = 1 + x + x^2 + x^3 + O(x^4).$$

We also know

$$\cos(x) = 1 - \frac{x^2}{2} + O(x^4).$$

Thus

$$\begin{aligned}
\frac{\cos(x)}{1 - x} &= \left(1 + x + x^2 + x^3 + O(x^4)\right)\left(1 - \frac{x^2}{2} + O(x^4)\right) \\
&= \left(1 + x + x^2 + x^3 + O(x^4)\right) - \frac{x^2}{2}\left(1 + x + x^2 + x^3 + O(x^4)\right) + O(x^4) \\
&= 1 + x + x^2 - \frac{x^2}{2} + x^3 - \frac{x^3}{2} + O(x^4) \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + O(x^4).
\end{aligned}$$

c. *We could again do long division, and again* ☹. *But let us again use the fact that*

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + O(x^5),$$

and thus

$$\begin{aligned}\frac{1}{1+x} = \frac{1}{1-(-x)} &= 1 + (-x) + (-x)^2 + (-x)^3 + (-x)^4 + O(x^5) \\ &= 1 - x + x^2 - x^3 + x^4 + O(x^5).\end{aligned}$$

Finally, let us plug in $x^2 + x^4$ *for* x *in the last series, so*

$$\frac{1}{1+x^2+x^4} = 1 - (x^2+x^4) + (x^2+x^4)^2 - (x^2+x^4)^3 + (x^2+x^4)^4 + O(x^{10}).$$

We know that

$$\begin{aligned}(x^2+x^4)^2 &= x^4 + 2x^6 + x^8, \\ (x^2+x^4)^3 &= x^6 + 3x^8 + O(x^{10}), \\ (x^2+x^4)^4 &= x^8 + O(x^{10}),\end{aligned}$$

and thus we have

$$\begin{aligned}\frac{1}{1+x^2+x^4} &= 1 - (x^2+x^4) + (x^4+2x^6+x^8) - (x^6+3x^8+O(x^{10})) + (x^8+O(x^{10})) + O(x^{10}) \\ &= 1 - x^2 - x^4 + x^4 + 2x^6 + x^8 - x^6 - 3x^8 + x^8 + O(x^{10}) \\ &= 1 - x^2 + x^6 - x^8 + O(x^{10}).\end{aligned}$$

d. *Products! No worries:*

$$\begin{aligned}\cos(x) - 1 &= -\frac{x^2}{2} + \frac{x^4}{24} + O(x^6), \\ \cos(x^2) - 1 &= -\frac{x^4}{2} + \frac{x^8}{24} + O(x^{12}), \\ \sin(x) &= x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7),\end{aligned}$$

so

$$\begin{aligned}x(\cos(x^2)-1)\sin(x) &= x\left(-\frac{x^4}{2} + \frac{x^8}{24} + O(x^{12})\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7)\right) \\ &= x\left(-\frac{x^4}{2}\left(x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7)\right) + \frac{x^8}{24}\left(x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7)\right)\right) + O(x^{12}) \\ &= x\left(-\frac{x^5}{2} + \frac{x^7}{12} - \frac{x^9}{240} + O(x^{11}) + \frac{x^9}{24} + O(x^{11})\right) + O(x^{12}) \\ &= -\frac{x^6}{2} + \frac{x^8}{12} - \frac{x^{10}}{240} + \frac{x^{10}}{24} + O(x^{12}) \\ &= -\frac{x^6}{2} + \frac{x^8}{12} + \frac{3x^{10}}{80} + O(x^{11}).\end{aligned}$$